Discrete Applied Mathematics

Volume 135, Issues 1–3, 15 January 2004, Pages 83–95

Residual reliability of P-threshold graphs ☆

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Abstract

We solve the problem of computing the residual reliability (the RES problem) for all classes of P-threshold graphs for which efficient structural characterizations based on decomposition to indecomposable components have been established. In particular, we give a constructive proof of existence of linear algorithms for computing residual reliability of pseudodomishold, domishold, matrogenic and matroidal graphs. On the other hand, we show that the RES problem is #P-complete on the class of biregular graphs, which implies the #P-completeness of the RES problem on the classes of indecomposable box-threshold and pseudothreshold graphs.

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1. Introduction

In this paper, we consider nondirected loopless graphs without multiple edges. Let *G* be a graph with the vertex set *VG* and the edge set *EG*; let *T*=*VG* or *T*=*EG*, and let $\mathscr{P}(T)$ be a family of some nonempty subsets of *T*. Then the pair $[T,\mathscr{P}(T)]$ is called a *graphoid system* with the support *T* and the path set $\mathscr{P}(T)$. If $\mathscr{P}(T)$ coincides with the set
$$\{X \mid X \subseteq T \text{ and there exists } Y \in \mathscr{P}(T) \text{ such that } Y \subseteq X\},$$
then the system $[T,\mathscr{P}(T)]$ is called *hierarchical*. Suppose that each element $s_i \in T$ can be excluded from *T* (i.e., can fail) independently from the others with probability $1-p_i$, where $0 \leqslant p_i \leqslant 1$. By $R(T,\mathscr{P}(T),\overline{p})$ denote the probability of the event that the set of remaining (working) elements is a path; here $\overline{p}=(p_1,\ldots,p_k)$ and $k=|T|$.

Assume that $\mathscr{P}(T)$ consists of all subsets of T that induce connected subgraphs of G. In this case, the quantities $R(EG, \mathscr{P}(EG), \bar{p})$ and $R_v(G, \bar{p}) = R(VG, \mathscr{P}(VG), \bar{p})$ are known as all-terminal and residual reliability of the graph G, respectively. Note that unlike other classical reliability criteria, the latter is defined for the system $[VG, \mathscr{P}(VG)]$ which is not hierarchical.

Algorithmic aspects of the problem of computing residual reliability (i.e., of the so-called RES problem) were studied in [1,5,9,10]. In these papers, polynomial algorithms were found for computing $R_v(G, \bar{p})$ of trees, interval graphs, and permutation graphs. On the other hand, the RES problem was proved to be #P-complete for split and bipartite graphs. [1]

In this paper, we solve the RES problem for those classes of P-threshold graphs whose efficient structural characterizations, based on decomposition to indecomposable components, were obtained in [2,4,11–13]. More specifically, we give a constructive proof of existence of linear algorithms computing residual reliability of matrogenic and pseudodomishold graphs. These results imply the existence of such algorithms for domishold, matroidal and threshold graphs. We also obtain efficient recurrence relations which make it possible to compute the coefficients of reliability polynomials for the above classes of graphs. On the other hand, we show that the RES problem is #P-complete even for biregular graphs with the probabilities of failures equal to $\frac{1}{2}$ for all vertices. This means that the RES problem is #P-complete in the classes of box-threshold and pseudothreshold graphs.

2. Basic definitions

All the notions of graph theory not defined here can be found in [6]. The definitions concerning P-threshold graphs are given according to [8].

Let R^+ be the set of nonnegative real numbers. A graph G is called *pseudodomishold* if for every induced subgraph H of G there exist a function $w_H : VH \to R^+$ not identical to zero and a number $t \in R^+$ such that for each subset $S \subseteq VH$ the following statements hold:

$$\begin{aligned} &\text{if } S \text{ is a dominating set in } H, \text{ then} \sum_{v \in S} w_H(v) \geqslant t, \\ &\text{if } S \text{ is not a dominating set in } H, \text{ then} \sum_{v \in S} w_H(v) \leqslant t. \end{aligned} \tag{1}$$

Replacing one of the inequalities in (1) by the strict inequality, we obtain the definition of a *domishold* graph.

A graph G is called *pseudothreshold* if there exists a function $w : VG \to R^+$ not identical to zero and a number $t \in R^+$ such that for each subset $S \subseteq VG$ the statements

[1] The notion of #P-completeness concerns enumeration problems (see, e.g. [7]).

hold

$$\begin{aligned} &\text{if } S \text{ is an independent set in } G, \text{ then} \sum_{v\in S} w(v) \leqslant t, \\ &\text{if } S \text{ is a dependent set in } G, \text{ then} \sum_{v\in S} w(v) \geqslant t. \end{aligned} \tag{2}$$

Replacing one of the inequalities in (2) by the strict inequality, we obtain the definition of a *threshold* graph. An equivalent definition can be given in terms of the quasi-order $\geqslant$: if $u, v \in VG$, then $v \geqslant u$ if and only if each vertex adjacent to u and different from v is adjacent to v.

A graph G is called *box-threshold* if every two vertices of G not comparable in the quasi-order $\geqslant$ have the same degree. A graph G is called *matrogenic* if the family of all subsets of VG inducing threshold graphs constitutes the independence system of a matroid with the support VG. Finally, G is called *matroidal* if the family of all subsets of EG such that the vertices incident to the edges of one subset induce threshold graphs constitutes the independence system of a matroid with the support EG.

If $p_1 = \cdots = p_n = p$ and $n = |VG|$, we define

$$R_v(G, p) = \sum_{i=0}^{n} c_i(G) p^i (1-p)^{n-i}, \tag{3}$$

where $c_i(G)$ is the number of connected induced subgraphs of G on i vertices. The expression in the right-hand side of (3) is called the (*residual*) *reliability polynomial* of the graph G with the coefficients $c_i(G)$ and will be denoted by $\mathrm{Pol}(G, p)$.

3. Intractable cases of the RES problem

In this section, the classical notions of #P-completeness and polynomial reducibility for enumeration problems are used; the details can be found in [7,14].

Let us start with several definitions. A graph G is called a *split graph* if there exists a partition $VG = A \cup B$ of its vertex set to a clique A and an independent set B. Moreover, if A and B are orbits in G (an orbit is a set of all vertices having the same degree d, and d is called the *degree of the orbit*), then G is called *biregular*.

A vertex v is *covered* by a matching P if v is incident with an edge of P. By $\mathscr{L}_k(G)$ denote the set of all matchings in G that consist of k edges; put $\mathscr{L}(G) = \bigcup_k \mathscr{L}_k(G)$ and $\mathrm{con}(G) = \sum_k c_k(G)$; the set of all perfect matchings in G is denoted by $\mathscr{E}(G)$. The problem of finding $|\mathscr{E}(G)|$ for the class of graphs with the set $\{d_1, \ldots, d_r\}$ of orbit degrees is denoted by $\mathrm{PM}(d_1, \ldots, d_r)$. Suppose that H is an induced subgraph in G and $P \in \mathscr{L}(H)$. We say that P is *induced* by a matching T from $\mathscr{L}(G)$ if $T \cap EH = P$. The *length* of a rational number p/q, where p and q are mutually prime integers, is defined to be $1 + \log_2(|p| + 1) + \log_2(|q| + 1)$.

Lemma 1 (Valiant [14]). *Let b be a vector of length n, A be a nonsingular $n \times n$ matrix, and the entries of A and b be rational numbers of lengths at most m. Then the system of linear equations $Ax = b$ can be solved in time polynomial on m and n.*

Theorem 1. *The RES problem is #P-complete in the class of biregular graphs with probabilities of all vertices equal to* $\frac{1}{2}$.

Proof. In [3], it was proved that the problem PM(2, 3) is #P-complete. We first show that PM(2, 3) is polynomially reducible to PM(3). Let G be a graph with the set of vertex degrees $\{2,3\}$ that contains a perfect matching. Since both $|VG|$ and the sum of degrees of all the vertices are even [6], it follows that the orbit of degree 2 in G consists of an even number of vertices, say $\{v_1,\dots,v_{2l}\}$. By F denote a simple cycle on $2l$ vertices with $VF=\{u_1,\dots,u_{2l}\}$, $VF\cap VG=\emptyset$. Let H be the graph with the vertex set $VH=VF\cup VG$ and the edge set

$$EH=EG\cup EF\cup\{u_iv_i\colon i=1,\dots,2l\}$$

(clearly, H is 3-regular). By $\mathscr{E}_k(H)$ denote the set of all perfect matchings in H containing k edges of F; put $g_k=|\mathscr{E}_k(H)|$. Since $|\mathscr{E}(F)|=2$ and $|\mathscr{E}_l(H)|=|\mathscr{E}(G)|\cdot|\mathscr{E}(F)|$, we have $2|\mathscr{E}(G)|=|\mathscr{E}_l(H)|$.

Replace each edge u_iu_{i+1} of EF in H by the subgraph M_{ir} shown in Fig. 1 (the sets $VM_{ir}\setminus\{u_i,u_{i+1}\}$ are assumed to be disjoint). The graph obtained is denoted by H_r. Suppose $S\in\mathscr{E}(H_r)$. It can be checked directly that the edges e_{01} and e_{02} either simultaneously belong to S or simultaneously do not. Therefore, the following surjection $\varphi:\mathscr{E}(H_r)\to\mathscr{E}(H)$ is well-defined: an edge $e=u_iu_{i+1}$ from EF belongs to $\varphi(S)$ if and only if $\{e_{01},e_{02}\}\subseteq S$; an edge e from $EH\setminus EF$ belongs to $\varphi(S)$ if and only if $e\in S$.

The graphs M_{ir} readily imply that:

$$(\text{either } e_{01}\notin S \text{ and } e_{03}\in S,\ \text{or } e_{01}\in S)\Rightarrow\begin{pmatrix}\text{either } \{e_{j2},e_{j4}\}\subseteq S,\\ \text{or } \{e_{j3},e_{j5}\}\subseteq S\\ \text{for every } j=1,\dots,r\end{pmatrix},$$

$$(e_{01}\notin S,\ e_{03}\notin S)\Rightarrow\left(S\cap EM_{ir}=\bigcup_{j=1}^{r}\{e_{j1},e_{j6},e_{j+1,1}\}\right).$$

Hence, we have

$$|\varphi^{-1}(\mathscr{E}_k(H))|=g_k(2^r)^k(2^r+1)^{2l-k}.$$

Consequently,

$$|\mathscr{E}(H_r)|=|\varphi^{-1}(\mathscr{E}(H))|=(2^r+1)^{2l}\sum_{k=0}^{l}\left(\frac{2^r}{1+2^r}\right)^k g_k$$

or

$$|\mathscr{E}(H_r)|/(2^r+1)^{2l}=\sum_{k=0}^{l}\left(\frac{2^r}{2^r+1}\right)^k g_k,\quad r=1,\dots,l+1. \tag{4}$$

The entries of the linear system (4) constitute a Vandermonde matrix. Thus it follows from Lemma 1 that the system (4) can be solved in time polynomial on l and $|EH_r|$. But, as shown above, $|\mathscr{E}(G)|=g_l/2$, which means that PM(2, 3) is polynomially reducible to PM(3).

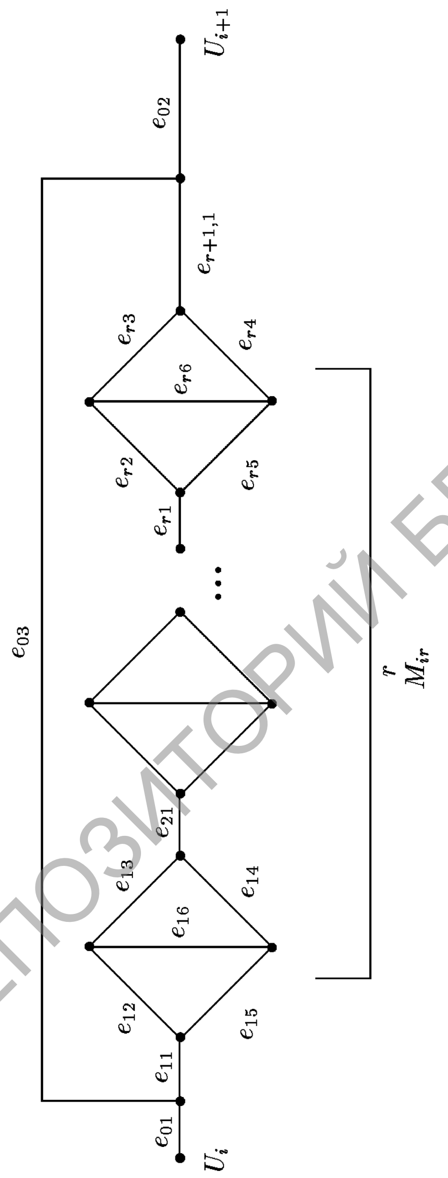

Fig. 1.

We now show that PM(3) is polynomially reducible to the problem IPReg of finding the number of all (not only perfect) matchings in the class of regular graphs. Let G be an arbitrary cubic graph and $VG=\{v_1,\ldots,v_n\}$. To each vertex v_i, we assign the graph L_{ir} shown in Fig. 2 by defining v_i to be adjacent to the vertices u_{r1} and u_{r2} of L_{ir} (the sets VL_{ir} are assumed to be disjoint). The graph obtained is denoted by F_r; clearly,

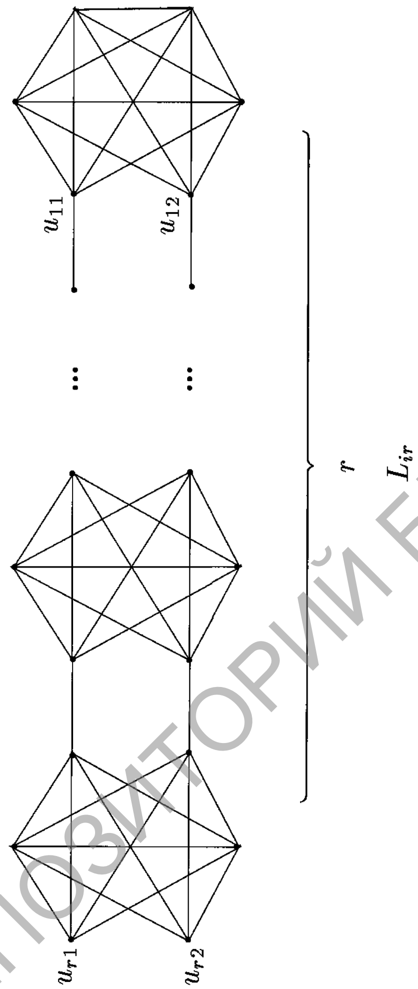

Fig. 2.

F_r is regular. By a_k, b_k, and c_k denote the number of matchings in L_{ik} covering both u_{k1} and u_{k2}, only u_{k1}, and neither u_{k1} nor u_{k2}, respectively. The following recurrence relations can be checked directly:

$$a_k = 22a_{k-1} + 56b_{k-1} + 36c_{k-1}, \quad a_1 = 24,$$

$$b_k = 14a_{k-1} + 40b_{k-1} + 28c_{k-1}, \quad b_1 = 16,$$

$$c_k = 8a_{k-1} + 24b_{k-1} + 18c_{k-1}, \quad c_1 = 10. \tag{5}$$

By N_{ir} denote the subgraph of F_r induced by the set of vertices $VG \cup VL_{ir}$. Suppose $S \in \mathscr{L}(G)$. If v_i is covered by the matching S, then S induces $|\mathscr{L}(L_{ir})|$ matchings in

N_{ir}, which number is equal to $a_r + 2b_r + c_r$. But if v_i is not covered by S, then S induces $a_r + 4b_r + 3c_r$ matchings in N_{ir}. Indeed, precisely $|\mathscr{L}(L_{ir})|$ of them do not contain $v_i u_{r1}, v_i u_{r2}$, and all the other $2b_r + 2c_r$ matchings contain only one of these edges. Put $m_i = |\mathscr{L}_i(G)|$. Then the total number of matchings in F_r is

$$|\mathscr{L}(F_r)| = \sum_{i=0}^{n/2} m_i (a_r + 2b_r + c_r)^{2i} (a_r + 4b_r + 3c_r)^{n-2i}.$$

Thus,

$$|\mathscr{L}(F_r)|/(a_r + 4b_r + 3c_r)^n = \sum_{i=0}^{n/2} m_i \left(\frac{a_r + 2b_r + c_r}{a_r + 4b_r + 3c_r} \right)^{2i}. \tag{6}$$

We next prove that there exists a constant r_0, which can be found in polynomial time, such that $d_i \neq d_j$ for all $i, j > r_0$, where

$$d_r = \frac{a_r + 2b_r + c_r}{a_r + 4b_r + 3c_r}.$$

Since

$$2\left(\frac{1}{d_r} - 1\right)^{-1} - 1 = \frac{a_r + b_r}{c_r + b_r} = \frac{a_r/b_r + 1}{c_r/b_r + 1},$$

it is sufficient to prove that the sequences $a_r \setminus b_r$ and $b_r \setminus c_r$ are strictly decreasing for $r > r_0$.

Let us introduce the following notation:

$$B = \begin{pmatrix} 22 & 56 & 36 \\ 14 & 40 & 28 \\ 8 & 24 & 18 \end{pmatrix}, \quad x_r = \begin{pmatrix} a_r \\ b_r \\ c_r \end{pmatrix}, \quad x_0 = \begin{pmatrix} 24 \\ 16 \\ 10 \end{pmatrix}.$$

It can be directly computed that the characteristic polynomial of the matrix $B/2$ is $\lambda^3 - 40\lambda^2 + 63\lambda - 8$, the eigenvalues of B are $\lambda_1 = 76.72648$, $\lambda_2 = 2.995$, and $\lambda_3 = 0.27854$, the eigenvectors of B are

$$\xi^1 = \begin{pmatrix} 2.3576 \\ 1.6611 \\ 1 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} -2.92146 \\ 0.3486 \\ 1 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 1.75144 \\ -1.32221 \\ 1 \end{pmatrix}$$

and the coefficients of the representation of x_0 in the base of ξ^1, ξ^2, ξ^3 are $\alpha_1 = 9.8564$, $\alpha_2 = -0.10937$, $\alpha_3 = 0.25297$. Since

$$x_r = B^r x_0 = B^r \sum_{i=1}^{3} \alpha_i \xi^i = \sum_{i=1}^{3} \alpha_i \lambda_i^r \xi^i,$$

we have

$$f(r) = \frac{a_r}{b_r} = \frac{\alpha_1 \xi_1^1 + \alpha_2 \xi_1^2 \gamma^r + \alpha_3 \xi_1^3 \Theta^r}{\alpha_1 \xi_2^1 + \alpha_2 \xi_2^2 \gamma^r + \alpha_3 \xi_2^3 \Theta^r},$$

where ξ_i^j is the ith entry of ξ^j, $\gamma = \lambda_2/\lambda_1$, and $\Theta = \lambda_3/\lambda_1$. Thus,

$$\frac{b_r^2 f'(r)}{\gamma^r} = \left(\alpha_2 \xi_1^2 \ln\gamma + \alpha_3 \xi_1^3 \ln\Theta \left(\frac{\lambda_3}{\lambda_2}\right)^r\right) b_r - a_r \left(\alpha_2 \xi_2^2 \ln\gamma + \alpha_3 \xi_2^3 \ln\Theta \left(\frac{\lambda_3}{\lambda_2}\right)^r\right). \tag{7}$$

Since $\lambda_3/\lambda_2 < 1/10$, (7) implies the existence of an integer r_1 such that $\operatorname{sgn}(f'(r)) = \operatorname{sgn}(\alpha_1\alpha_2\xi_1^2\xi_2^1 \ln\gamma - \alpha_1\alpha_2\xi_2^2\xi_1^1 \ln\gamma) < 0$ for $r > r_1$.

Analogously, we prove that there exists an integer r_2 such that $r > r_2$ implies $\operatorname{sgn}(g'(r)) = \operatorname{sgn}(\alpha_1\alpha_2\xi_3^1\xi_2^2 \ln\gamma - \alpha_1\alpha_2\xi_3^2\xi_2^1 \ln\gamma) < 0$, where $g(r) = b_r/c_r$. Putting $r_0 = \max\{r_1, r_2\}$ completes the proof.

So, we have proved that the entries of the system (6) consitute a Vandermonde matrix for $r = r_0+1, \ldots, r_0+1+n/2$. Therefore, due to Lemma 1, the system can be solved in time polynomial on $|EF_r|$ and n. This means the polynomial reducibility of PM(3) to IPReg since $m_{n/2} = |\mathscr{E}(G)|$.

It remains to prove that IPReg is polynomially reducible to the RES problem in the class of biregular graphs with the probability of vertex failure equal to $\frac{1}{2}$. To do so, consider a regular graph G and its line graph F and put $m = |VF|$, $n = |EF|$. By $\mathscr{M}_k(F)$ denote the set of vertex subsets of F covering exactly k edges; put $t_k = |\mathscr{M}_k(F)|$. By the definition of line graphs, F is regular. Let S be a subset of edges in EG, and T be the corresponding vertex subset in F. It can be easily checked that $T \in \mathscr{M}_n(F)$ if and only if $EG \setminus S \in \mathscr{L}(G)$. Thus, $t_n = |\mathscr{L}(G)|$. Add to F the n r-element sets V_{ir} consisting of new 2-degree vertices and join each vertex in V_{ir} to the two end vertices of the ith edge in F. Then we augment F with edges to obtain the complete subgraph on the vertex set VF. Denote the graph obtained by H_r. Clearly, H_r is a biregular graph.

Let $C(H_r)$ be the set of all connected induced subgraphs in H_r containing some vertices in VF (there are $nr = |\bigcup_i V_{ir}|$ more connected induced one-vertex subgraphs in H_r). Define the mapping $\varphi : C(H_r) \to 2^{VF}$ such that $\varphi(M) = VF \cap VM$ and pick an edge $e_i \in EF$. If e_i is not covered by $\varphi(M)$, then since M is connected, we have $VM \cap V_{ir} = \emptyset$. On the other hand, if e_i is covered with $\varphi(M)$, then each subset of V_{ir} (including the empty one) can belong to VM. Consequently,

$$|\varphi^{-1}(\mathscr{M}_k(F))| = t_k (2^r)^k, \quad |C(H_r)| = \sum_{k=0}^{n} t_k 2^{rk}$$

or

$$\operatorname{con}(H_r) - nr = \sum_{k=0}^{n} t_k 2^{rk}, \quad r = 1, \ldots, n+1$$

(recall that $\operatorname{con}(H_r)$ is the number of all connected induced subgraphs of H_r). Now Lemma 1 implies the polynomial reducibility of IPReg to the problem of determining the number of connected induced subgraphs in biregular graphs. But

$$\frac{\operatorname{con}(H_r)}{2^{m+nr}} = \operatorname{Pol}\left(H_r, \frac{1}{2}\right),$$

which completes the proof of Theorem 1.

Since each biregular graph is a box-threshold graph [13], Theorem 1 directly implies

Theorem 2. *The RES problem is #P-complete in the class of box-threshold graphs with the probabilities of failure equal to $\frac{1}{2}$ for all vertices.*

Since each biregular graph is pseudothreshold, we have

Corollary 1. *The RES problem is #P-complete in the class of pseudothreshold graphs.*

4. Polynomially solvable cases of the RES problem

A graph G with an ordered partition $VG = A \cup B$ of the vertex set is called a *triad* and denoted by (G,A,B). The set of triads and the set of all graphs will be denoted by $\mathscr{P}$ and $\mathscr{I}$, respectively. The composition $\circ : \mathscr{P} \times \mathscr{I} \to \mathscr{I}$ is defined as follows: if $VG \cap VH = \emptyset$, $G \in \mathscr{P}$, and $H \in \mathscr{I}$, then

$$(G,A,B) \circ H = G \cup H \cup K_{A,VH},$$

where $K_{A,VH}$ is the complete bipartite graph with the parts A and VH. Moreover, if $H \in \mathscr{P}$ then

$$(G,A,B) \circ (H,C,D) = ((G,A,B) \circ H,\ A \cup C,\ B \cup D).$$

The composition $\circ$ is an associative operation [12].

A triad (G,A,B) is called a *net-graph* if $|A| = |B|$, G is a split-graph with parts A and B, and the set of edges obtained from EG by deleting all the edges from A is a perfect matching in G. The vertices $a \in A$ and $b \in B$ in a net-graph (G,A,B) (or in its complement) will be called *corresponding* if a and b are adjacent in G. Let $\bar{G}$ be the complement of G; then the triad $\bar{F} = (\bar{G},A,B)$ is called *complement* to the triad $F = (G,A,B)$.

Now introduce some notation: $O_n = (G,\emptyset,B)$, where $n = |B|$ and B is an independent set in G; $Q_n = (G,A,\emptyset)$, where $n = |A|$ and A is an independent set in G; $\mathscr{M}$ is the set of net-graphs and their complements; $P_4 = (G,A,B)$, where $|A| = |B| = 2$ and $(G,A,B) \in \mathscr{M}$; $\mathscr{R}$ is the set of complements of cycles and paths; P_5 and C_5 are the path and the cycle on 5 vertices, respectively. Suppose (G,A,B) is a triad. Then $u_k(G)$ is the number of k-vertex induced subgraphs whose each component contains at least one vertex of A, and $P_u(G,\bar{p})$ is the probability for the components of the subgraph induced by the remaining vertices to possess this property. Finally, p_v is the probability for the vertex v to remain, $P_f(G,\bar{p}) = \prod_{v \in VG}(1 - p_v)$, and

$$\mathrm{Upol}(G,p) = \sum_{k=0}^{n} u_k(G)p^k(1-p)^{n-k}, \quad n = |VG|.$$

Lemma 2 (Chernyak [2] and Chernyak and Chernyak [4]). *A graph G is pseudodomishold if and only if G can be decomposed as*

$$G = X_1 \circ \cdots \circ X_m \circ Y_1 \circ \cdots \circ Y_n \circ Z, \quad m \geqslant 0, \quad n \geqslant 0,$$

where $X_i \in \{Q_2, Q_1, O_1, P_4\}$, $1 \leqslant i \leqslant m$, $Y_j = (F_j, A_j, \emptyset)$, $F_j \in \mathscr{R}$, $1 \leqslant j \leqslant n$, *and* $Z \in \mathscr{R} \cup \{P_5\}$; *this decomposition can be done in time* $\mathrm{O}(|VG| + |EG|)$.

Lemma 3 (Tyshkevich [11]). *A graph G is matrogenic if and only if G can be decomposed as*

$$G = X_1 \circ \cdots \circ X_m \circ Y_1 \circ \cdots \circ Y_n$$

or

$$G = X_1 \circ \cdots \circ X_m \circ Z,$$

where $X_i \in \{Q_1, O_1\} \cup \mathscr{M}$, $1 \leqslant i \leqslant m$, $Y_1 = \cdots = Y_n = Y$, $Y \in \{Q_2, \overline{O_2}\}$, *and* $Z \in \{C_5\}$; *this decomposition can be done in time* $\mathrm{O}(|VG| + |EG|)$.

Lemma 4. *Let* $G \in \mathscr{P}$. *Then*

$$R_v = (G \circ H, \bar{p}) = R_v(H, \bar{p}) \cdot P_f(G, \bar{p}) + R_v(G, \bar{p}) \cdot P_f(H, \bar{p}) + P_u(G, \bar{p})(1 - P_f(H, \bar{p})). \tag{8}$$

Proof. Let F be a connected induced subgraph of $G \circ H$. Then $F = (F_1, A, B) \circ F_2$, where F_1 and F_2 are induced subgraphs of G and H, respectively. If $VF_1 = \emptyset$ ($VF_2 = \emptyset$), then F_2 (F_1) is a connected induced subgraph of H (G), and this situation is taken into account in the first two addends of (8). Suppose that $VF_i \neq \emptyset$, $i = 1, 2$. Then since F is connected, each component of F_1 must contain a vertex from A. In this case, F_2 is an arbitrary nonempty induced subgraph of H. This situation is taken into account in the third addend of (8). The lemma is proved.

Lemma 5. *Let* $G \in \mathscr{P}$, $m = |VG|$, *and* $n = |VH|$. *Then*

$$c_k(G \circ H) = c_k(H) + c_k(G) + \sum_{i=1}^{k-1} u_i(G) \binom{n}{k-i}, \quad 1 \leqslant k \leqslant m + n. \tag{9}$$

Proof. Due to Lemma 4, we have

$$\mathrm{Pol}(G \circ H, p) = \mathrm{Pol}(H, p)(1-p)^m + \mathrm{Pol}(G, p)(1-p)^n + \mathrm{Upol}(G, p)(1 - (1-p)^n).$$

Using this fact and the equality

$$1 - (1-p)^n = \sum_{i=1}^{n} \binom{n}{i} p^i (1-p)^{n-i},$$

we obtain (9). The lemma is proved. □

Lemma 6. *If A is a clique in* (G, A, B), *we have*

$$R_v(G \circ H, \bar{p}) = R_v(H, \bar{p}) \cdot P_f(G, \bar{p}) + R_v(G, \bar{p}),$$

$$c_k(G \circ H) = c_k(H) + \sum_{i=1}^{k} c_i(G) \binom{n}{k-i}, \quad 1 \leqslant k \leqslant m + n.$$

Proof. If A is a clique in the triad (G,A,B), then each disconnected induced subgraph of G has a component disjoint with A. Therefore, $P_u(G,\bar{p}) = R_v(G,\bar{p})$, and the first equality follows from Lemma 4. The second formula follows from the equality

$$c_k(G) + \sum_{i=1}^{k-1} c_i(G)\binom{n}{k-i} = \sum_{i=1}^{k} c_i(G)\binom{n}{k-i}. \qquad \square$$

Lemma 7. *Suppose (G,A,B) is a triad. Then $B=\emptyset$ implies*

$$R_v(G\circ H,\bar{p}) = R_v(H,\bar{p})\cdot P_f(G,\bar{p}) + R_v(G,\bar{p})\cdot P_f(H,\bar{p}) + (1-P_f(G,\bar{p}))(1-P_f(H,\bar{p})),$$

$$c_k(G\circ H) = c_k(H) + c_k(G) + \binom{m+n}{k} - \binom{m}{k} - \binom{n}{k},$$

and $A=\emptyset$ implies

$$R_v(G\circ H,\bar{p}) = R_v(H,\bar{p})\cdot P_f(G,\bar{p}) + R_v(G,\bar{p})\cdot P_f(H,\bar{p}),$$

$$c_k(G\circ H) = c_k(H) + c_k(G).$$

This lemma follows directly from Lemmas 4 and 5.

In Lemma 8, all indices are equal to $1,\ldots,n$ modulo n.

Lemma 8. *Let $H\in\mathcal{R}$ and $VH=\{1,\ldots,n\}$, where $n\geqslant 5$, and the vertices be numbered so that the vertices i and $i+1$ are not adjacent whenever $1\leqslant i\leqslant n$. Then*

$$R_v(H,\bar{p}) = 1 - \sum_{i=1}^{l} p_i p_{i+1} p_{i+2} \prod_{j\notin\{i,i+1,i+2\}} (1-p_j) - \sum_{i=1}^{k} p_i p_{i+1} \prod_{j\notin\{i,i+1\}} (1-p_j),$$

$$c_k(H) = \binom{n}{k}, \quad k\neq 2,3, \quad c_2(H) = \binom{n}{2} - k, \quad c_3(H) = \binom{n}{3} - l,$$

where $l=k=n$ if $\bar{H}$ is a cycle and $l=n-2$, $k=n-1$ if $\bar{H}$ is a path.

Proof. Let F be a disconnected induced subgraph of H. Since the degrees of vertices in $\bar{F}$ are at most 2, the components of F have one or two vertices. Moreover, since $\bar{F}$ does not contain triangles, there are at most two such components. If F consisted of two components having two vertices each, then $\bar{F}$ would contain C_4 as an induced subgraph, which is impossible. Thus, either $VF=\{i,i+1\}$ or $VF=\{i,i+1,i+2\}$. The rest of the proof is obvious.

Lemma 9. *Suppose (G,A,B) is a net-graph with $A=\{1,\dots,n\}$ and $B=\{n+1,\dots,2n\}$ such that the vertices i and $n+i$ are corresponding for all $1\leqslant i\leqslant n$. Then*

$$R_v(G,\bar p)=\prod_{i=1}^{n}(1-(1-p_i)p_{n+i})-\prod_{i=1}^{2n}(1-p_i)+\sum_{i=n+1}^{2n}p_i\prod_{j\neq i}(1-p_j),$$

$$R_v(\bar G,\bar p)=1-\prod_{i=n+1}^{2n}(1-p_i)-\sum_{i=1}^{n}p_ip_{n+i}\prod_{\substack{j\neq n+i,\\ j=n+1}}^{2n}(1-p_j), \tag{10}$$

$$c_1(G)=c_1(\bar G)=2n$$

and for all $k>1$,

$$c_k(G)=\sum_{i=0}^{n}\binom{n}{i}\binom{i}{k-i},\quad c_k(\bar G)=\binom{2n}{k}-\binom{n}{k}-n\binom{n-1}{k-2}.$$

Proof. Let F be an induced subgraph in G. Then F is connected if and only if either $|VF|=1$, or $VF\neq\emptyset$ and the implication

$$(n+i\in VF)\Rightarrow(i\in VF),$$

holds for each vertex $i\in A$. The probability of this event is defined by (10). So, putting $t=p/(1-p)$, we have

$$\begin{aligned}\mathrm{Pol}(G,p)&=(1-p+p^2)^n-(1-p)^{2n}+np(1-p)^{2n-1}\\&=(1-p)^{2n}((t^2+t+1)^n-1+nt)\\&=(1-p)^{2n}\left(\sum_{i=0}^{n}\binom{n}{i}t^i\sum_{j=0}^{i}\binom{i}{j}t^j-1+nt\right).\end{aligned}$$

Thus, for every $k>1$,

$$c_k(G)=\sum_{i+j=k}\binom{n}{i}\binom{i}{j}.$$

Now let F be a disconnected induced subgraph in $\bar G$. Since any two vertices in B constitute a dominating set in $\bar G$, we have $|VF\cap B|\leqslant 1$. If, in addition, $n+i\in VF\cap B$, then $i\in VF\cap A$. The rest of the proof is obvious.

Theorem 3. *For the classes of pseudodomishold and matrogenic graphs the RES problem can be solved in linear time.*

Proof. Let a graph G be an either pseudodomishold or matrogenic. Due to Lemmas 2 and 3, we can decompose G as $G=H_r\circ\cdots\circ H_2\circ H_1$ in $O(|VG|+|EG|)$ time, where

$$H_i\in\{Q_2,Q_1,O_1,\overline{O_2},P_4,P_5,C_5,\mathscr{M},\mathscr{R}\},\quad 1\leqslant i\leqslant r.$$

Moreover, if $i\geqslant 2$ then $H_i=(F_i,A_i,B_i)$ and either A_i is a clique, or $A_i=\emptyset$, or else $B_i=\emptyset$ (all these cases were considered in Lemmas 6 and 7). Due to Lemmas 8 and 9, the value of $R_v(H_i, \bar{p})$, $1 \leqslant i \leqslant r$, can be computed in time linear with respect to the order of H_i (if $|VH_i| \leqslant 5$ then $R_v(H_i, \bar{p})$ can be computed directly from the definition). But then due to Lemmas 6 and 7 the values $R_v(H_j \circ \cdots \circ H_1, \bar{p})$, $2 \leqslant j \leqslant r$, can be computed successively in time linear with respect to the order of G. Theorem 3 is proved. □

Since all matroidal graphs are matrogenic, and all domishold and threshold graphs are pseudodomishold, Theorem 3 implies the following corollaries:

Corollary 2. *The RES problem can be solved in the classes of matroidal and domishold graphs in linear time.*

Corollary 3 (Stivaros [9]). *The RES problem can be solved in the class of threshold graphs in linear time.*

Theorem 4. *The coefficients of the reliability polynomial of a pseudodomishold (matrogenic, domishold, matroidal, threshold) graph G can be found in* $O(|VG|^3)$ *time.*

The proof of Theorem 4 is similar to that of Theorem 3.

References

[1] F. Boesch, A. Satyanarayana, C.L. Suffel, On residual connectedness network reliability, in: American Mathematical Society, Reliability of Computer and Communication Networks, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 5, Providence, RI, 1991, pp. 51–59.

[2] A.A. Chernyak, Pseudodomishold graphs, Vesci NAN Belarusi, Ser. Fiz.-Mat. Nauk 4–5 (1988) 37–43 and 18–22 (in Russian).

[3] A.A. Chernyak, Dual reliability problems for k-uniform hypergraphs, Dokl. NAN Belarusi 43 (2) (1999) 34–35 (in Russian).

[4] A.A. Chernyak, Zh.A. Chernyak, Pseudodomishold graphs, Discrete Math. 84 (1990) 193–196.

[5] C.J. Colbourn, A. Satyanarayana, C. Suffel, K. Sutner, Computing residual connectedness reliability for restricted networks, Discrete Appl. Math. 44 (1993) 221–232.

[6] F. Harary, Graph Theory, Addison–Wesley, 1969.

[7] M.R. Garey, D.S. Johnson, Computers and Intractability—A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.

[8] N.R. Mahadev, U.N. Peled, Threshold Graphs and Related Topics, North-Holland, Amsterdam, 1995.

[9] C. Stivaros, On the residual node connectedness network reliability model, Ph.D. Dissertation, Stevens Institute of Technology, 1990.

[10] K. Sutner, A. Satyanarayana, C. Suffel, The complexity of the residual node connectedness reliability problem, SIAM J. Comput. 20 (1993) 149–155.

[11] R.I. Tyshkevich, Once more on matrogenic graphs, Discrete Math. 51 (1984) 91–100.

[12] R.I. Tyshkevich, A.A. Chernyak, Graph decomposition, Kibernetika 2 (1985) 67–74 (in Russian).

[13] R.I. Tyshkevich, A.A. Chernyak, Boxthreshold graphs: structure and enumeration, in: Graphen und Netzwerke—Theorie und Anwendungen, Proceedings of the 30 International Wissenschaftliches Kolloquium (Ilmenau, 1985), Technische Hoohschule Ilmenau, Ilmenau, 1985, pp. 119–121.

[14] L. Valiant, The complexity of enumeration and reliability problems, SIAM J. Comput. 8 (3) (1979) 410–421.