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On the Tits alternative for some generalized triangle groups

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Abstract. One says that the Tits alternative holds for a finitely generated group Γ if Γ contains either a non abelian free subgroup or a solvable subgroup of finite index. Rosenberger states the conjecture that the Tits alternative holds for generalized triangle groups $T(k, l, m, R) = \langle a, b; a^k = b^l = R^m(a, b) = 1\rangle$. In the paper Rosenberger's conjecture is proved for groups $T(2, l, 2, R)$ with $l = 6, 12, 30, 60$ and some special groups $T(3, 4, 2, R)$.

Introduction

J. Tits [15] proved that if G is a finitely generated linear group then G contains either a non abelian free subgroup or a solvable subgroup of finite index. Let Γ be an arbitrary finitely generated group. One says that the Tits alternative holds for Γ if Γ satisfies one of these conditions.

An one-relator free product of a family of groups $\{G_i\}$, $i \in I$, is called the group $G = (*G_i)/N(S)$, where S is a cyclically reduced word in the free product $*G_i$, $N(S)$ is its normal closure. S is called the relator. One-relator free products share many properties with one-relator groups [7]. We consider the case when G_i's are cyclic groups.

Definition 1. *A group Γ having a presentation*

$$\Gamma =< a_1, \ldots, a_n; a_1^{l_1} = \ldots = a_n^{l_n} = R^m(a_1, \ldots, a_n) = 1 >, \qquad (1)$$

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where $n \geq 2$, $m \geq 1$, $l_i = 0$ or $l_i \geq 2$ for all i, $R(a_1, \ldots, a_n)$ is a cyclically reduced word in the free group on $a_1, \ldots, a_n$ which is not a proper power, is called an one-relator product of n cyclic groups.

One relator products of cyclic groups provide a natural algebraic generalization of Fuchsian groups which are one relator products of cyclics relative to the standard Poincare presentation (see [6])

$$F = \langle a_1, \ldots, a_p, b_1, \ldots, b_t, c_1, d_1, \ldots, c_g, d_g; \\ a_i^{m_i} = a_1 \ldots a_p b_1 \ldots b_t [c_1, d_1] \ldots [c_g, d_g] = 1 \rangle.$$

If $n = 2$ and $m \geq 2$ then we have so-called *generalized triangle groups*

$$T(k, l, m, R) = \langle a, b; a^k = b^l = R^m(a, b) = 1 \rangle.$$

If $R(a, b) = ab$ then we obtain an ordinary triangle group.

Let Γ be a group of the form (1) and $m \geq 2$. If either $n \geq 4$ or $n = 3$ and $(l_1, l_2, l_3) \neq (2, 2, 2)$ then Γ contains a free subgroup of rank 2 [5]. If $n = 3$ and $(l_1, l_2, l_3) = (2, 2, 2)$ then Γ either contains a free subgroup of rank 2 or a free abelian subgroup of rank 2 and index 2.

The case when Γ is a generalized triangle group is much more difficult. Rosenberger stated the following conjecture.

Conjecture 1 ([13]). *The Tits alternative holds for generalized triangle groups.*

Fine, Levin, and Rosenberger proved this conjecture in the following cases: 1) $l = 0$ or $k = 0$; 2) $m \geq 3$ [5]. Now suppose that $k, l, m \geq 2$. Let $s(\Gamma) = 1/k + 1/l + 1/m$. If $s(\Gamma) < 1$ then Baumslag, Morgan and Shalen [1] proved that the group Γ contains a non abelian free subgroup. Using some new methods, Howie [8] proved Conjecture 1 in the case $s(\Gamma) = 1$ and up to equivalence $R \neq ab$. If $s(\Gamma) = 1$ and $R = ab$ then Γ is an ordinary triangle group. The classical result says that Γ contains $\mathbb{Z}$ as a subgroup of finite index.

Now consider groups of the form

$$\Gamma = T(2, l, 2, R) = \langle a, b; a^2 = b^l = R^2(a, b) = 1 \rangle, \tag{2}$$

where $l > 2$, $R = ab^{v_1} \ldots ab^{v_s}$, $0 < v_i < l$. In the following cases Conjecture 1 holds for Γ: 1) $s \leq 4$ [13], [9]; 2) $l > 5$ and $l \neq 6, 10, 12, 15, 20, 30, 60$ [2], [3]. In this paper we prove two theorems.

Theorem 1. *Let Γ be a group of the form (2) with $s \geq 5$ and $l \in \{6, 12, 30, 60\}$. Then Γ contains a free subgroup of rank 2.*

Theorem 2. *Let* $\Gamma = \langle a, b; a^3 = b^4 = R^2(a,b) = 1\rangle$, *where* $R = a^{u_1}b^{v_1}\dots a^{u_s}b^{v_s}$ *with* $0 < u_i < 3$ *and* $0 < v_i < 4$. *In the following cases* Γ *contains a non-abelian free subgroup: i)* $V = \sum_{i=1}^{s} v_i$ *is even; ii)* s *is even.*

Thus, Conjecture 1 is still open for groups $T(2, l, 2, R)$ with $l = 3, 4, 5, 10, 15, 20$ and groups $T(3, l, 2, R)$ with $l = 3, 4, 5$.

1. Some auxiliary results

In this section we prove several auxiliary results used in the proofs of theorems 1 and 2. Throughout we shall denote the ring of algebraic integers in $\mathbb{C}$ by $\mathcal{O}$, the group of units in $\mathcal{O}$ by $\mathcal{O}^*$, the free group of a rank 2 with generators g and h by $F_2 =< g, h >$, the greatest common divisor of integers a and b by (a, b). the image of a matrix $A \in \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{PSL}_2(\mathbb{C})$ by $[A]$, the trace of a matrix A by $\mathrm{tr}\, A$, the identity matrix in $\mathrm{SL}_2(\mathbb{C})$ by E. The following lemma characterizes elements of finite order in $\mathrm{PSL}_2(\mathbb{C})$.

Lemma 1. *Let* $2 \le m \in \mathbb{Z}$ *and* $\pm E \neq X \in \mathrm{SL}_2(\mathbb{C})$. *Then* $[X]^m = 1$ *in* $\mathrm{PSL}_2(\mathbb{C})$ *if and only if* $\mathrm{tr}\, X = 2\cos\frac{r\pi}{m}$ *for some* $r \in \{1, \dots, m-1\}$.

The proof easily follows from the fact that $\varepsilon, \varepsilon^{-1}$, where ε is a root of unity of degree m, are the eigenvalues of the matrix X.

We shall use standard facts from geometric representation theory (see [4, 10]). Here we recall some notations. Let $F_n = \langle g_1, \dots, g_n\rangle$ be a free group, $R(F_n) = \mathrm{SL}_2(\mathbb{C})^n$ be a representation variety of F_n in $\mathrm{SL}_2(\mathbb{C})$ The group $\mathrm{GL}_2(\mathbb{C})$ acts naturally on $R(F_n)$ (by simultaneous conjugation of components) and its orbits are in one-to-one correspondence with the equivalence classes of representations of F_n. Under this action orbits of $\mathrm{GL}_2(\mathbb{C})$ are not necessarily closed and so the variety of orbits (the geometric quotient) is not an algebraic variety. However one can consider the categorical quotient $R(F_n)/\mathrm{GL}_2(\mathbb{C})$ (see [12]), which we shall denote by $X(F_n)$ and call the variety of characters. By construction, its points parametrize closed $\mathrm{GL}_2(\mathbb{C})$-orbits. It is well known that an orbit of a representation is closed iff the corresponding representation is fully reducible and so the points of the variety $X(F_n)$ are in one-to-one correspondence with the equivalence classes of fully reducible representations of Γ in $\mathrm{SL}_2(\mathbb{C})$.

For an arbitrary element $g \in F_n$ one can consider the regular function

$$\tau_g : R(F_n) \to \mathbb{C}, \qquad \tau_g(\rho) = \mathrm{tr}\,\rho(g).$$

Usually, τ_g is called *a Fricke character* of the element g. It is known that the $\mathbb{C}$-algebra $T(F_n)$ generated by all functions τ_g, $g \in F_n$, is equal to $\mathbb{C}[X(F_n)] = \mathbb{C}[R(F_n)]^{\mathrm{GL}_2(\mathbb{C})}$. Combining results of [4, 14] it is easy to see that $T(F_n)$ is generated by Fricke characters $\tau_{g_i} = x_i$, $\tau_{g_i g_j} = y_{ij}$, $\tau_{g_i g_j g_k} = z_{ijk}$, where $1 \le i < j < k \le n$. Consider a morphism $\pi : R(F_n) \to \mathbb{A}^t$ defined by

$$\pi(\rho) = (x_1(\rho), \ldots, x_n(\rho), y_{12}(\rho), \ldots, y_{n-1,n}(\rho), \\ z_{123}(\rho), \ldots, z_{n-2,n-1,n}(\rho)), \quad (3)$$

where $t = n + n(n-1)/2 + n(n-1)(n-2)/6$. The image $\pi(R(F_n))$ is closed in $\mathbb{A}^t$ [4]. Since $X(F_n)$ and $\pi(R(F_n))$ are biregularly isomorphic, we shall identify $X(F_n)$ and $\pi(R(F_n))$. Obviously, $\dim R(F_n) = 3n$, $\dim X(F_n) = 3n - 3$. Set

$$R^s(F_n) = \{\rho \in R(F_n) \mid \rho \text{ is irreducible}\}, \qquad X^s(F_n) = \pi(R^s(F_n)).$$

$R^s(F_n)$, $X^s(F_n)$ are open in Zariski topology subsets of $R(F_n)$, $X(F_n)$ respectively [4].

Now, consider a free group $F_2 = \langle g, h \rangle$. The ring $T(F_2)$ is generated by the functions $\tau_g, \tau_h, \tau_{gh}$.

Lemma 2. *For all $\alpha, \beta, \Gamma \in \mathbb{C}$ there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that $\tau_g(A,B) = \operatorname{tr} A = \alpha$, $\tau_h(A,B) = \operatorname{tr} B = \beta$, $\tau_{gh}(A,B) = \operatorname{tr} AB = \Gamma$.*

This lemma can be easily proved by straightforward computations.

Lemma 2 implies that $X(F_2) = \pi(R(F_2)) = \mathbb{A}^3$. Moreover, the functions $\tau_g, \tau_h, \tau_{gh}$ are algebraically independent over $\mathbb{C}$ and for every $u \in F_2$ we have

$$\tau_u = Q_u(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_u \in \mathbb{Z}[x, y, z]$ is a uniquely determined polynomial with integer coefficients [4]. The polynomial Q_u is usually called the Fricke polynomial of the element u.

Consider polynomials $P_n(\lambda)$ satisfying the initial conditions $P_{-1}(\lambda) = 0$, $P_0(\lambda) = 1$ and the recurrence relation

$$P_n(\lambda) = \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda).$$

If $n < 0$ then we set $P_n(\lambda) = -P_{|n|-2}(\lambda)$. The degree of the polynomial $P_n(\lambda)$ is equal to n if $n > 0$ and to $|n| - 2$ if $n < 0$. It is easy to verify by induction on n that

$$P_n(2\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi}. \quad (4)$$

It follows from (4) that the polynomial $P_n(\lambda)$, $n \geq 1$, has n zeros described by the formula

$$\lambda_{n,k} = 2\cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \ldots, n. \tag{5}$$

Moreover, it is easy to verify by induction that for $n \geq 0$ we have

$$\begin{aligned} P_{2n}(\lambda) &= \lambda^{2n} + \cdots + (-1)^n \\ P_{2n-1}(\lambda) &= \lambda(\lambda^{2n-2} + \cdots + (-1)^{n-1}n). \end{aligned} \tag{6}$$

Lemma 3. *Let $k, l \in \mathbb{Z}$, $(k,l) = 1$ and $l \geq 2$ is not a power of a prime. Then $2\sin\frac{k\pi}{l} \in \mathcal{O}^*$.*

Proof. Let $l = 2^t u$, where u is odd. If $t = 1$ then k is odd and $2\sin\frac{k\pi}{l} = 2\cos\frac{r\pi}{u}$ with $r = (u-k)/2 \in \mathbb{Z}$ Since $u - 1$ is even, it follows from (6) that $2\cos\frac{r\pi}{u} \in \mathcal{O}^*$.

If $t > 1$ then k is odd and $2\sin\frac{k\pi}{l} = 2\cos\frac{r\pi}{2^t u}$ with $r = 2^{t-1}u - k$.

If $t = 0$ then $2\sin\frac{k\pi}{l} = 2\cos\frac{r\pi}{2u}$ with $r = u - 2k$.

Thus, it is sufficient to prove that $2\cos\frac{r\pi}{2^t u} \in \mathcal{O}^*$, where $t \geq 1$, $(r, 2^t u) = 1$, $u > 1$ and u is not a power of a prime in the case $t = 1$. Let $u = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$, where p_i is a prime and $0 < \alpha_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, s$. By (5) numbers $\lambda_i = 2\cos\frac{i}{2^t u}\pi$, $i = 1, 2, \ldots, 2^t u - 1$, are the roots of the polynomial $P_{2^t u-1}(\lambda)$, so that

$$P_{2^t u-1}(\lambda) = \prod_{i=1}^{2^t u-1} (\lambda - \lambda_i)$$

and the constant term of $P_{2^t u-1}$ is equal to $(-1)^{2^{t-1}-1} 2^{t-1} p_1^{\alpha_1} \ldots p_s^{\alpha_s}$. On the other hand, the polynomials $P_{2p_i^{\alpha_i}-1}(\lambda)$, i=1,2,. . . ,s, and $P_{2^t-1}(\lambda)$ has the roots $2\cos\frac{j\pi}{2p_i^{\alpha_i}}$, $j = 1, 2, \ldots, 2p_i^{\alpha_i} - 1$, and $2\cos\frac{j\pi}{2^t}$, $j = 1, 2, \ldots, 2^t - 1$, respectively. Hence, all these polynomials divide $P_{2^t u-1}(\lambda)$ and any two of them have only one common root $\lambda = 0$. Hence,

$$P_{2^t u-1}(\lambda) = F(\lambda)F_1(\lambda),$$

where

$$F(\lambda) = \lambda^{-s} P_{2^t-1}(\lambda) \prod_{i=1}^{s} P_{2p_i^{\alpha_i}-1}(\lambda).$$

By (5) the constant term of $F(\lambda)$ is equal to $(-1)^{2^{t-1}-1} 2^{t-1} p_1^{\alpha_1} \ldots p_s^{\alpha_s}$. Consequently, the constant term and the leading coefficient of $F_1(\lambda)$ are equal to 1. Since $2\cos\frac{r\pi}{2^t u}$ is not a root of $F(\lambda)$, it is a root of $F_1(\lambda)$ and we obtain $2\cos\frac{r\pi}{2^t u} \in \mathcal{O}^*$ as required. ∎

Furthermore, we require the more detailed information on the Fricke polynomials. Let $w = g^{\alpha_1}h^{\beta_1}\dots g^{\alpha_s}h^{\beta_s} \in F_2$ and let $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. Let us treat the Fricke polynomial $Q_w(x,y,z)$ as a polynomial in z. Set

$$Q_w(x,y,z) = M_n(x,y)z^n + M_{n-1}(x,y)z^{n-1} + \dots + M_0(x,y).$$

Lemma 4 ([16]). *The degree of the Fricke polynomial $Q_w(x,y,z)$ with respect to z is equal to s and its leading coefficient $M_s(x,y)$ has the form*

$$M_s(x,y) = \prod_{i=1}^{s} P_{\alpha_i-1}(x))P_{\beta_i-1}(y). \tag{7}$$

A subgroup $H \in \mathrm{PSL}_2(\mathbb{C})$ is called *non-elementary* if H is infinite, irreducible and non-isomorphic to a dihedral group.

Lemma 5 ([11]). *Let $H \in \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary subgroup. Then H contains a non-abelian free subgroup.*

Lemma 6 ([4]). *Let $A, B \in \mathrm{SL}_2(\mathbb{C})$ and $\operatorname{tr} A = x$, $\operatorname{tr} B = y$, $\operatorname{tr} AB = z$. A subgroup $< A, B >$ is irreducible if and only if*

$$\operatorname{tr} ABA^{-1}B^{-1} = x^2 + y^2 + z^2 - xyz - 2 \neq 2.$$

2. Proof of Theorem 1.

Let Γ be a group from Theorem 1, that is,

$$\Gamma = T(2,l,2,R) = \langle a, b; a^2 = b^l = R^2(a,b) = 1\rangle, \tag{8}$$

where $R = ab^{v_1}\dots ab^{v_s}$, $0 < v_i < l$, $s > 4$. Set $V = \sum_{i=1}^{s} v_i$. If $(V,l) \neq 1$ then Γ contains a non-abelian free subgroup (see [2]). So we shall assume that $(V,l) = 1$. To prove Theorem 1, we construct a representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ such that $\rho(\Gamma)$ contains a non-abelian free subgroup. Let k be an integer such that $\frac{k}{l} = \frac{k'}{l'}$ with $(k', l') = 1$ and $l' > 5$. Set

$$\beta_k = 2\cos\frac{k\pi}{l}, \qquad f_{R,k}(z) = Q_R(0, \beta_k, z), \tag{9}$$

where Q_R is the Fricke polynomial of R.

Definition 2. *Let z_0 be a root of a polynomial $f_{R,k}(z)$ and $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr} A = 0$, $\operatorname{tr} B = \beta_k$, $\operatorname{tr} AB = z_0$. We shall denote by $G(z_0)$ a subgroup of $\mathrm{PSL}_2(\mathbb{C})$, generated by $[A], [B]$.*

The group $G(z_0)$ is an epimorphic image of Γ since by Lemma 1

$$[A]^2 = [B]^l = R^2([A],[B]) = 1.$$

Lemma 7. *Numbers* $\pm 2\sin\frac{k\pi}{l}$ *are not roots of the polynomial* $f_{R,k}(z)$*.*

Proof. Suppose that $f_{R,k}(-2\sin\frac{k\pi}{l}) = 0$. Let ε be a primitive root of unity of degree $2l$. Consider a representation $\rho_k : F_2 \to \mathrm{SL}_2(\mathbb{C})$ defined by

$$\rho_k(g) = A = \begin{pmatrix} \varepsilon^{l/2} & 0 \\ 1 & \varepsilon^{-l/2} \end{pmatrix}, \qquad \rho_k(h) = B_k = \begin{pmatrix} \varepsilon^k & x \\ 0 & \varepsilon^{-k} \end{pmatrix}. \tag{10}$$

Then we have $\operatorname{tr} A = 0$, $\operatorname{tr} B_k = \beta_k$, $\operatorname{tr} AB_k = x - 2\sin\frac{k\pi}{l}$. So we obtain

$$f_{R,k}(z)(\rho_k) = f_{R,k}(x - 2\sin\frac{k\pi}{l}) = g_k(x) = \operatorname{tr} R(A, B_k).$$

Since $-2\sin\frac{k\pi}{l}$ is a root of $f_{R,k}(z)$, 0 is a root of $g_k(x)$. This means that a constant term of $g_k(x)$ is equal to 0. On the other hand, a constant term of $\operatorname{tr} R(A, B_{-k})$ is equal to

$$\varepsilon^{ls/2+kV} + \varepsilon^{-ls/2-kV} = 2\cos(\frac{ls/2+kV}{l}\pi) \neq 0,$$

since $(V,l) = 1$ by assumption. This contradiction proves that $2\sin\frac{k\pi}{l}$ is not a root of $f_{R,k}(z)$. Analogously, considering a matrix B_{-k} instead the matrix B_k, we obtain that $2\sin\frac{k\pi}{l}$ is not a root of $f_{R,k}(z)$. ☐

Lemma 8. *Assume that the polynomial* $f_{R,k}(z)$ *has a root* $z_0 \neq 0$*. Then* Γ *contains a non-abelian free subgroup.*

Proof. By Lemma 7 we have $z_0 \neq \pm 2\sin\frac{k\pi}{l}$. Let us show that $G(z_0)$ is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. First, $G(z_0)$ is irreducible by Lemma 6 since

$$\operatorname{tr} ABA^{-1}B^{-1} - 2 = z_0^2 - 4\sin^2\frac{k\pi}{l} \neq 0.$$

Second, $G(z_0)$ is not a dihedral group since two of three numbers $\operatorname{tr} A$, $\operatorname{tr} B$, $\operatorname{tr} AB$ are not equal to 0 (see [11]). Third, it follows from classification of finite subgroups of SLC [11] that $G(z_0)$ is infinite since it is irreducible and contains an element $[B]$ of order > 5. Thus, $G(z_0)$ (and consequently Γ) contains a non-abelian free subgroup. ☐

Bearing in mind Lemmas 7 and 8, we shall assume in what follows that

$$f_{R,k}(z) = M_{R,k}z^s, \tag{11}$$

where by lemma 4

$$M_{R,k} = \prod_{i=1}^{s} P_{v_i-1}(2\cos\frac{k\pi}{l}) = (2\sin\frac{k\pi}{l})^{-s}\prod_{i=1}^{s} 2\sin\frac{v_ik\pi}{l}. \tag{12}$$

Lemma 9. *In the following cases Γ contains a non-abelian free subgroup:*

1) $l = 6$, s is odd and there exists i such that $v_i \in \{2, 3, 4\}$;

2) $l = 6$, s is even and either there exists i such that $v_i = 3$ or there exist i, j such that $i \neq j$ and $v_i, v_j \in \{2, 4\}$;

3) $l > 6$ and there exists i such that 6 divides v_i.

Proof. Let $f_{R,k}(z) = M_{R,k}z^s$ and ρ_{-k} be a representation defined by (10). Then

$$g_k(x) = f_{R,k}(x + 2\sin\frac{k\pi}{l}) = M_{R,k}(x + 2\sin\frac{k\pi}{l}) = \operatorname{tr} R(A, B_{-k}). \tag{13}$$

Comparing constant terms in (13),we obtain

$$\prod_{i=1}^{s} 2\sin\frac{v_ik\pi}{l} = 2\cos\frac{ls/2 - kV}{l}\pi. \tag{14}$$

1) If $l = 6$, $s = 2s_1 + 1$ then we set $k = 1$ and obtain $2\cos\frac{6s_1+3-V}{6}\pi = \pm 1$ since $(V, 6) = 1$. Suppose that there exists i such that $v_i \in \{2, 3, 4\}$. Then

$$\delta = P_{v_i-1}(2\cos\frac{\pi}{6}) = \frac{2\sin v_i\pi/6}{2\sin\pi/6} \in \{\sqrt{3}, 2\}$$

and we have from (14)

$$\prod_{j=1}^{s} P_{v_j-1}(2\cos\frac{\pi}{6}) = \delta\prod_{j\neq i} P_{v_j-1}(2\cos\frac{\pi}{6}) = \pm 1. \tag{15}$$

It follows from (15) that $1/\delta \in \mathcal{O}$ which is a contradiction.

2) If $l = 6$ and $s = 2s_1$ then we set $k = 1$ and obtain $2\cos\frac{6s_1-V}{6}\pi = \pm\sqrt{3}$ since $(V, 6) = 1$. First, suppose that there exists i such that $v_i = 3$. Then

$$P_{v_i-1}(2\cos\frac{\pi}{6}) = \frac{2\sin v_i\pi/6}{2\sin\pi/6} = 2$$

and we have from (14)

$$\prod_{j=1}^{s} P_{v_j-1}(2\cos(\frac{\pi}{6})) = 2\prod_{j\neq i} P_{v_j-1}(2\cos(\frac{\pi}{6})) = \pm\sqrt{3}. \tag{16}$$

It follows from (16) that $\sqrt{3}/2 \in \mathcal{O}$ which is a contradiction.

Now, suppose that there exists i, j such that $v_i, v_j \in \{2, 4\}$. For $r \in \{i, j\}$ we have

$$P_{v_r-1}(2\cos\frac{\pi}{6}) = \frac{2\sin v_r\pi/6}{2\sin\pi/6} = \sqrt{3}.$$

Hence by (14)

$$\prod_{k=1}^{s} P_{v_k-1}(2\cos\frac{\pi}{6}) = 3\prod_{k\neq i, k\neq j} P_{v_k-1}(2\cos\frac{\pi}{6}) = \pm\sqrt{3}. \qquad (17)$$

It follows from (17) that $\sqrt{3}/3 \in \mathcal{O}$ which is a contradiction.

3) If $l \in \{12, 30\}$ then by assumptions of the lemma there exists i such that $v_i = 6$. Set $k = 1$. Then

$$2\sin\frac{v_i\pi}{l} = \begin{cases} 2, & \text{if } l = 12, \\ 2\sin\frac{\pi}{5} = \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{2}, & \text{if } l = 30. \end{cases}$$

In both cases $2\sin\frac{v_i\pi}{l} \notin \mathcal{O}^*$. On the other hand, $2\cos\frac{ls/2-V}{l}\pi \in \mathcal{O}^*$ by lemma (3) and (14) implies

$$\frac{1}{2\sin\frac{v_i\pi}{l}} = \frac{1}{2\cos\frac{ls/2-V}{l}\pi}\prod_{j\neq i} 2\sin\frac{v_j\pi}{l} \in \mathcal{O},$$

which is a contradiction.

If $l = 60$ and there exists i such that $v_i = 30$ then we set $k = 1$. As before we obtain from (14) that $2\sin\frac{v_i\pi}{60} = 2 \in \mathcal{O}^*$ which is a contradiction. If for any i we have $v_i \neq 30$ then we set $k = 2$ and obtain a contradiction in the same way as in the case $l = 30$. □

Let A, B_k be matrices defined in (10), $W(A, B_k) = AB_k^{u_1} \dots AB_k^{u_s}$, where $0 < u_i < l$. A set $(u_1, \dots, u_s)$ will be considered as cyclically ordered. Let

$$l_i = |\{j \mid u_j = i\}|, \qquad f_{i,j} = |\{r \mid u_r = i, u_{r+1} = j\}|. \qquad (18)$$

We have following equations:

$$\sum_{i=1}^{l-1} l_i = s, \quad \sum_{i=1}^{l-1} f_{ij} = l_j, \quad \sum_{j=1}^{l-1} f_{ij} = l_i, \quad i, j = 1, \dots, l-1. \qquad (19)$$

Lemma 10. *Let* $g(x) = \operatorname{tr} W(A, B_t) = a_0 x^s + \cdots + a_s$*,* $h_i = P_{i-1}(\varepsilon^k + \varepsilon^{-k})$*. Then we have* $a_0 = \prod_{j=1}^{s} h_{u_j}$ *and*

$$\begin{aligned} a_2 =& a_0 \sum_{j=1}^{l-1} \frac{f_{ii}}{h_i} \left(\frac{l_i - 2}{h_i} + \sum_{j \neq i} \frac{l_j \varepsilon^{ti-tj}}{h_j} \right) + \\ & a_0 \sum_{i \neq j} \frac{f_{ij}}{h_i} \left(\frac{l_i - 1}{h_i} + \frac{(l_j - 1)\varepsilon^{ti-tj}}{h_j} + \sum_{k \neq i, k \neq j} \frac{l_k \varepsilon^{ti-tk}}{h_k} \right) - \\ & a_0 \left(\sum_{i=1}^{l-1} \frac{l_i(l_i - 1)}{2h_i^2} (\varepsilon^{2ti} + \varepsilon^{-2ti}) + \sum_{i \neq j} \frac{l_i l_j}{h_i h_j} (\varepsilon^{ti+tj} + \varepsilon^{-ti-tj}) \right). \end{aligned} \tag{20}$$

This lemma can be proved by direct computations.

2.1. The case $l = 6$, s is odd.

Bearing in mind Lemma 9, we have $v_i \in \{1, 5\}$ for every i. Set $k = 1$ and $M_R = M_{R,1}$. Then $M_R = \prod_{i=1}^{s} P_{v_i - 1}(2\cos\frac{\pi}{6}) = 1$ since $P_0 = 1$ and $P_4(2\cos\frac{\pi}{6}) = \frac{2 sin 5\pi/6}{2\sin\pi/6} = 1$. Consequently,

$$f_R(z) = z^s. \tag{21}$$

Consider a representation $\rho : F_2 \to \mathrm{PSL}_2(\mathbb{C})$, $\rho(g) = A$, $\rho(h) = B_1$, where A, B_1 are defined in (10). Then we have

$$f_1(x) = \operatorname{tr} R(A, B_1) = (x - 1)^s. \tag{22}$$

Further, the equations (19) have the form

$$\begin{aligned} &f_{11} + f_{15} = l_1, \qquad && f_{11} + f_{51} = l_1, \qquad && l_1 + l_5 = s, \\ &f_{55} + f_{15} = l_5, && f_{55} + f_{51} = l_5. \end{aligned} \tag{23}$$

It follows from (23) that $f_{15} = f_{51}$. Taking into account Lemma 10, we obtain that the coefficient by x^{s-2} of the polynomial $f_1(x)$ is equal to

$$\begin{aligned} a_2 = & f_{11}(l_1 - 2 + l_5\varepsilon^{-4}) + f_{15}(l_1 - 1 + (l_5 - 1)\varepsilon^{-4}) + \\ & f_{51}((l_1 - 1)\varepsilon^4 + l_5 - 1) + f_{55}(l_1\varepsilon^4 + l_5 - 2) - \\ & \frac{l_1(l_1 - 1)}{2} - \frac{l_5(l_5 - 1)}{2} + 2l_1 l_5 = 3f_{15} + \frac{s^2}{2} - \frac{3}{2}s. \end{aligned} \tag{24}$$

On the other hand, $a_2 = s(s-1)/2$ by (22). Thus, we obtain

$$s = 3f_{15}. \tag{25}$$

Now, consider an epimorphic image $\Gamma_1 = \langle c, d; c^2 = d^3 = R^2(c,d) = 1\rangle$ of the group Γ, where $R(c,d) = cd^{v_1}\dots cd^{v_s}$. We can write the word $R(c,d)$ from the free product $\langle c; c^2 = 1\rangle * \langle d; d^3 = 1\rangle$ in the form $R_1(c,d) = cd^{u_1}\dots cd^{u_s}$, where $u_i = \begin{cases} 1, & \text{if } v_i = 1, \\ 2, & \text{if } v_i = 5. \end{cases}$ Let $U = \sum_{i=1}^{s} u_i$. Since $(V, 6) = 1$, we have $(U, 3) = 1$. Set

$$P(z) = Q_{R_1}(0, 1, z),$$

where Q_{R_1} is a Fricke polynomial of R_1.

Lemma 11. *If the polynomial $P(z)$ has a root z_0 which is not equal to 0, ± 1, $\pm\sqrt{2}$, $\frac{\pm 1 \pm\sqrt{5}}{2}$, $\pm\sqrt{3}$ then the group Γ_1 (and, consequently, Γ) contains a non-abelian free subgroup.*

Proof. Let $X, Y \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr} X = 0$, $\operatorname{tr} Y = 1$, $\operatorname{tr} XY = z_0$. Let $H = \langle [X], [Y]\rangle \subset \mathrm{PSL}_2(\mathbb{C})$. First, H is infinite (see [17]). Second, H is not dihedral group since $[Y]$ has order 3. Third, H is irreducible since $\operatorname{tr} XYX^{-1}Y^{-1} - 2 = z_0^2 - 3 \neq 0$. Thus, H is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Consequently, H contains a non-abelian free subgroup. □

Since the polynomial $P(z)$ has integer coefficients and bearing in mind Lemma 11, we may assume that $P(z)$ has the form

$$P(z) = z^{\alpha_1}(z^2-1)^{\alpha_2}(z^2-2)^{\alpha_3}(z^2-z-1)^{\alpha_4}(z^2+z-1)^{\alpha_5}(z^2-3)^{\alpha_6}. \quad (26)$$

Consider a representation $\delta : F_2 \to \mathrm{SL}_2(\mathbb{C})$, $g \mapsto A$, $h \mapsto B_2$, where A, B_2 are defined in (10). We have $\operatorname{tr} A = 0$, $\operatorname{tr} B_2 = 1$, $\operatorname{tr} AB_2 = x - \sqrt{3}$. Consequently,

$$\begin{aligned} P_1(x) = \tau_{R_1}(0,1,z)(\delta) = P(x-\sqrt{3}) = (x-\sqrt{3})^{\alpha_1}(x^2 - 2\sqrt{3}x + 2)^{\alpha_2} \\ \cdot (x^2 - 2\sqrt{3}x + 1)^{\alpha_3}(x^2 - (2\sqrt{3}+1)x + 2 + \sqrt{3})^{\alpha_4} \\ \cdot (x^2 - (2\sqrt{3}-1)x + 2 - \sqrt{3})^{\alpha_5}(x - 2\sqrt{3})^{\alpha_6}x^{\alpha_6} = \operatorname{tr} R_1(A, B_2). \quad (27) \end{aligned}$$

The constant term of the polynomial $\operatorname{tr} R_1(A, B_2)$ is equal to

$$\varepsilon^{3s+2U} + \varepsilon^{-3s-2U} = 2\cos\frac{3s+2U}{6}\pi = \pm\sqrt{3}$$

since s is odd and $(U, 3) = 1$. Comparing constant terms in (27), we obtain $\alpha_6 = 0$ and

$$(-\sqrt{3})^{\alpha_1}2^{\alpha_2}(2+\sqrt{3})^{\alpha_4}(2-\sqrt{3})^{\alpha_5} = \pm\sqrt{3}. \quad (28)$$

It follows from (28) that $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_4 = \alpha_5$. Thus, the polynomial $P_1(x)$ has the form:

$$P_1(x) = (x-\sqrt{3})(x^2-2\sqrt{3}x+1)^{\alpha_3}(x^4-4\sqrt{3}x^3+15x^2-6\sqrt{3}x+1)^{\alpha_4}. \quad (29)$$

In particular,

$$2\alpha_3 + 4\alpha_4 + 1 = s. \quad (30)$$

It follows from (29) that the coefficient of $P_1(x)$ by x^{s-2} is equal to

$$a_2 = \frac{3}{2}s^2 - \frac{5}{2}s + 1 + \alpha_4. \quad (31)$$

On the other hand, we have by Lemma 10

$$a_2 = f'_{11}(l'_1 - 2 + l'_2\varepsilon^{-2}) + f'_{12}(l'_1 - 1 + (l'_2 - 1)\varepsilon^{-2}) + f'_{21}((l'_1 - 1)\varepsilon^2 + l'_2 - 1) + f'_{22}(l'_1\varepsilon^2 + l'_2 - 2) + \frac{l'_1(l'_1 - 1)}{2} + \frac{l'_2(l'_2 - 1)}{2} + 2l'_1 l'_2 = f'_{12} + \frac{3}{2}s^2 - \frac{5}{2}s, \quad (32)$$

where $f'_{11} = f_{11}$, $f'_{12} = f_{15}$, $f'_{21} = f_{51}$, $f'_{22} = f_{55}$, $l'_1 = l_1$, $l'_2 = l_5$. It follows from (31), (32) that

$$f_{15} = 1 + \alpha_4. \quad (33)$$

Equations (25), (30), and (33) imply

$$2\alpha_3 + \frac{s}{3} - 3 = 0. \quad (34)$$

Since $\alpha_3 \geq 0$, it follows from (34) that $\frac{s}{3} - 3 \leq 0$, that is, $s \leq 9$. Thus, if $s > 9$ then either $f_R(z)$ is not of the form (21) or $P(z)$ is not of the form (26). Bearing in mind lemmas 8 and 11, we obtain that if $l = 6$, s is odd and $s > 9$ then Γ contains a non-abelian free subgroup.

Now, let $s \leq 9$. Since $s > 4$, s is odd and $s = 3f_{15}$ by (25), we must have $s = 9$, $f_{15} = 3$. Furthermore, without loss of generality we can assume $l_1 > l_5$. Moreover, one can cyclically shift the sequence $(v_1, \dots, v_s)$. This transformation replaces the relation $R^2(a,b)$ with an equivalent one. It is easy to see that there exists only 9 words R under these conditions:

$$\begin{aligned} &R_1 = abababab ab^5abab^5abab^5, && R_2 = abababab^5ababab^5abab^5,\\ &R_3 = abababab^5abab^5ababab^5, && R_4 = abababab^5ab^5abab^5abab^5,\\ &R_5 = abababab^5abab^5abab^5ab^5, && R_6 = abababab^5abab^5ab^5abab^5,\\ &R_7 = ababab^5ab^5ababab^5abab^5, && R_8 = ababab^5ab^5abab^5ababab^5,\\ &R_9 = ababab^5ababab^5abab^5ab^5. && \end{aligned} \quad (35)$$

Direct computations show that $f_{R_i}(z) \neq z^9$ for $i = 1, \dots, 7$. But

$$f_{R_8}(z) = f_{R_9}(z) = z^9.$$

Since $R_9(a, b)$ is conjugate to $R_8(a^{-1}, b^{-1})^{-1}$, it is sufficient to consider only the group $\Gamma = \langle a, b; a^2 = b^6 = R_8^2(a, b) = 1\rangle$.

Lemma 12. *The group* Γ *contains a non-abelian free subgroup.*

Proof. Consider a dihedral group $D_3 = \langle c, d; c^2 = d^2 = (cd)^3 = 1\rangle$ of order 6 and a homomorphism

$$\psi : \Gamma \to D_3, \qquad a \mapsto c,\ b \mapsto d.$$

Obviously, $\psi(R_8) = 1$, that is, ψ is well defined and ψ is an epimorphism. Let $\Gamma_1 = \ker\psi \subset \Gamma$. Then $[\Gamma : \Gamma_1] = 6$. Using Reidemeister–Schreier rewriting process, we obtain that Γ_1 has a presentation of the form

$$\Gamma_1 = \langle g_1, g_2, g_3, g_4; g_1^3 = g_2^3 = (g_3g_4)^3 = (g_3^2g_4^{-1})^2 = (g_3^{-1}g_4^2)^2 = W_1^2(g_1, g_2, g_4) = W_1^2(g_2, g_1, g_3) = W_2^2(g_1, g_2, g_3) = W_2^2(g_2, g_4, g_1) = 1\rangle, \quad (36)$$

where $W_1(g, h, t) = tgh^2tgh^2th^2$, $W_2(g, h, t) = t^{-1}gt^{-1}gt^{-1}gh^2$.

To prove Lemma 12, it is sufficient to construct a representation $\delta : \Gamma_1 \to \mathrm{PSL}_2(\mathbb{C})$ such that the group $\delta(\Gamma_1)$ is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Let us consider matrices

$$A_1 = \begin{pmatrix} x_1 & \frac{-x_1^2+x_1-1}{y_1} \\ y_1 & 1-x_1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} i & -1 \\ 0 & -i \end{pmatrix},$$
$$A_2 = \begin{pmatrix} x_2 & \frac{-x_2^2+x_2-1}{y_2} \\ y_2 & 1-x_2 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then we have $\operatorname{tr} A_1 = \operatorname{tr} A_2 = \operatorname{tr} A_3A_4 = 1$, $\operatorname{tr} A_3^2A_4^{-1} = \operatorname{tr} A_3^{-1}A_4^2 = 0$. Therefore,

$$[A_1]^3 = [A_2]^3 = ([A_3][A_4])^3 = ([A_3]^2[A_4]^{-1})^2 = ([A_3]^{-1}[A_4])^2 = 1$$

by Lemma 1. Let us suppose that the following conditions hold:

$$\operatorname{tr} A_1A_3 = \operatorname{tr} A_2A_4 = \sqrt{2}, \qquad \operatorname{tr} A_2A_3 = \operatorname{tr} A_1A_4, \quad (37)$$

$$\operatorname{tr} W_1(A_1, A_2, A_4) = \operatorname{tr} W_1(A_2, A_1, A_3) = \operatorname{tr} W_2(A_1, A_2, A_3) = \operatorname{tr} W_2(A_2, A_4, A_1) = 0 \quad (38)$$

It follows from (37) that

$$x_2 = \frac{3x_1^2 + (-2 + 3i\sqrt{2})x_1 - i\sqrt{2} - 4/3}{2x_1 + i\sqrt{2} - 1}, \qquad y_1 = 2ix_1 - \sqrt{2} - i,$$
$$y_2 = \frac{3ix_1^2 - (2\sqrt{2} + 3i)x_1 + \sqrt{2} + i/3}{2x_1 + i\sqrt{2} - 1}. \tag{39}$$

Substituting (39) in (38), one obtains

$$\operatorname{tr} W_1(A_1, A_2, A_4) = \operatorname{tr} W_1(A_2, A_1, A_3) = \frac{h_1(x_1)}{(2x_1 + i\sqrt{2} - 1)^4},$$
$$\operatorname{tr} W_2(A_1, A_2, A_3) = \operatorname{tr} W_2(A_2, A_4, A_1) = \frac{h_2(x_1)}{(2x_1 + i\sqrt{2} - 1)^2}, \tag{40}$$

where

$$h_1(x_1) = -24i + \frac{137\sqrt{2}}{9} - \left(\frac{184i}{3} + \frac{424\sqrt{2}}{3}\right) x_1 + \left(\frac{1790i}{3} + 22\sqrt{2}\right) x_1^2 + (-329i + 683\sqrt{2})x_1^3 - (975i + 446\sqrt{2})x_1^4 + (648i - 420\sqrt{2})x_1^5 + (198i + 261\sqrt{2})x_1^6 + (-108i + 18\sqrt{2})x_1^7 - 9\sqrt{2}x_1^8,$$

$$h_2(x_1) = 3\sqrt{2} + 4i/3 + (4\sqrt{2} - 16i)x_1 + (-10\sqrt{2} + 18i)x_1^2 + (-9\sqrt{2} + 3i)x_1^3 - 3ix_1^4.$$

One can check that h_2 devides h_1. Let x_1' be a roort of the equation $h_2(x_1) = 0$ and let x_2', y_1', y_2' be defined by (39). Then the set $\{x_1', x_2', y_1', y_2'\}$ is a solution of equations (37), (38). Hence, matrices A_1, A_2, A_3, A_4 define a required representation

$$\delta : \Gamma_1 \to \mathrm{PSL}_2(\mathbb{C}), \qquad \delta(g_i) = [A_i],\ i = 1, 2, 3, 4.$$

Let us show that $\delta(\Gamma_1)$ is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Consider a subgroup $G = \langle [A_1A_3], [A_2A_4] \rangle \subset \delta(\Gamma_1)$. By construction, we have $\operatorname{tr} A_1A_3 = \operatorname{tr} A_2A_4 = \sqrt{2}$. Next,

$$\operatorname{tr} A_1A_3A_2A_4 = \frac{h_3(x_1')}{(2x_1' + i\sqrt{2} - 1)^2} = \Delta,$$

where

$$h_3(x_1') = -3x_1'^4 + (6 - 6\sqrt{2}i)x_1'^3 + (11 - 9\sqrt{2}i)x_1'^2 + (-14 + 5\sqrt{2}i)x_1' - 4\sqrt{2}i - 1/3.$$

Direct computations show that $\Delta \notin \{0,1,2\}$. By Lemma 6, G is irreducible and infinite (see [17]). Obviously, G is not a dihedral group. Therefore, G (and consequently Γ_1) is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. $\square$

2.2. The case $l = 6$, s is even.

Since $(6, u) = 1$ and bearing in mind Lemma 9, we can assume without loss of generality that

$$R = ab^{v_1} \dots ab^{v_s},$$

where $v_1 \in \{2, 4\}$, $v_i \in \{1, 5\}$ for $i = 2, \dots, s$. Moreover, we can assume that $v_1 = 2$ applying otherwise to the word R an automorphism $b \mapsto b^{-1}$ of a cyclic group $\langle b; b^2 = 1\rangle$. Thus, $M_R = \prod_{i=1}^{s} P_{v_i-1}(2\cos\frac{\pi}{6}) = \sqrt{3}$ since $P_0 = 1$, $P_4(2\cos\frac{\pi}{6}) = \frac{2sin(5\pi/6)}{2\sin(\pi/6)} = 1$, and $P_1(2\cos\frac{\pi}{6}) = 2\cos(\frac{\pi}{6}) = \sqrt{3}$. Taking into account Lemma 8, we shall assume that

$$f_R(z) = \sqrt{3}z^s.$$

Further, the equations (19) have the form

$$\begin{aligned} &f_{11} + f_{12} + f_{15} = l_1, \quad &&f_{15} + f_{25} + f_{55} = l_5, \quad &&f_{12} + f_{52} = 1,\\ &f_{11} + f_{21} + f_{51} = l_1, \quad &&f_{51} + f_{52} + f_{55} = l_5, \quad &&f_{21} + f_{25} = 1,\\ &\qquad l_1 + l_5 = s - 1. \end{aligned} \tag{41}$$

It follows from (41) that

$$\begin{aligned} &f_{11} = l_1 - f_{12} - f_{15}, \quad f_{55} = s - l_1 - 2 - f_{15} + f_{21}, \qquad f_{25} = 1 - f_{21},\\ &f_{51} = f_{12} + f_{15} - f_{21}, \quad l_5 = s - l_1 - 1, \quad f_{52} = 1 - f_{12}. \end{aligned} \tag{42}$$

Consider a representation $\rho : F_2 \to \mathrm{PSL}_2(\mathbb{C})$, $\rho(g) = A$, $\rho(h) = B_1$, where A and B_1 are defined by (10). Then we have

$$f_1(x) = \mathrm{tr}\, R(A, B_1) = \sqrt{3}(x-1)^s. \tag{43}$$

Bearing in mind Lemma 10 and (42), we obtain that the coefficient by x^{s-2} of the polynomial $f_1(x)$ is equal to

$$a_2 = \sqrt{3}\left(\frac{1}{2}s^2 + \frac{1}{2}s + 2 - 2f_{21} + f_{12} + 3f_{15}\right). \tag{44}$$

On the other hand, $a_2 = \sqrt{3}s(s-1)/2$. Thus, we obtain

$$s + 2f_{21} - f_{12} - 3f_{15} - 2 = 0. \tag{45}$$

Now, consider an epimorphic image Γ_1 of the group Γ:

$$\Gamma_1 = \langle c, d; c^2 = d^3 = R^2(c, d) = 1\rangle,$$

where $R(c, d) = cd^{v_1} \dots cd^{v_s}$. We can write the word $R(c, d)$ from the free product $\langle c; c^2 = 1\rangle * \langle d; d^3 = 1\rangle$ in the form $R_1(c, d) = cd^{u_1} \dots cd^{u_s}$, where $u_i = \begin{cases} 1, & \text{if } v_i = 1, \\ 2, & \text{if } v_i = 5 \text{ or } v_i = 2. \end{cases}$ Let $U = \sum_{i=1}^{s} u_i$. Since $(V, 6) = 1$, we have $(U, 3) = 1$. Set

$$P(z) = Q_{R_1}(0, 1, z),$$

where Q_{R_1} is a Fricke polynomial of R_1. Since the polynomial $P(z)$ has integer coefficients and bearing in mind Lemma 11, we can assume that $P(z)$ has the form

$$P(z) = \sqrt{3}z^{\alpha_1}(z^2-1)^{\alpha_2}(z^2-2)^{\alpha_3}(z^2-z-1)^{\alpha_4}(z^2+z-1)^{\alpha_5}(z^2-3)^{\alpha_6}. \quad (46)$$

Consider a representation $\delta : F_2 \to \mathrm{SL}_2(\mathbb{C})$, $g \mapsto A$, $h \mapsto B_2$. We have $\operatorname{tr} A = 0$, $\operatorname{tr} B_2 = 1$, $\operatorname{tr} AB_2 = x - \sqrt{3}$. Consequently,

$$\begin{aligned} P_1(x) = Q_{R_1}(0, 1, z)(\delta) = P(x - \sqrt{3}) = (x - \sqrt{3})^{\alpha_1}(x^2 - 2\sqrt{3}x + 2)^{\alpha_2} \\ \cdot (x^2 - 2\sqrt{3}x + 1)^{\alpha_3}(x^2 - (2\sqrt{3} + 1)x + 2 + \sqrt{3})^{\alpha_4} \\ \cdot (x^2 - (2\sqrt{3} - 1)x + 2 - \sqrt{3})^{\alpha_5}(x - 2\sqrt{3})^{\alpha_6}x^{\alpha_6} = \operatorname{tr} R_1(A, B_2). \quad (47) \end{aligned}$$

The constant term of the polynomial $\operatorname{tr} R_1(A, B_2)$ is equal to

$$\varepsilon^{3s+2U} + \varepsilon^{-3s-2U} = 2\sin\left(\frac{3s + 2U}{6}\pi\right) = \pm 1$$

since s is even and $(U, 3) = 1$. Comparing constant terms in (47), we obtain $\alpha_6 = 0$ and

$$(-\sqrt{3})^{\alpha_1}2^{\alpha_2}(2 + \sqrt{3})^{\alpha_4}(2 - \sqrt{3})^{\alpha_5} = \pm 1. \quad (48)$$

It follows from (48) that $\alpha_1 = \alpha_2 = 0$, $\alpha_4 = \alpha_5$. Thus, the polynomial $P_1(x)$ has the form:

$$P_1(x) = (x^2 - 2\sqrt{3}x + 1)^{\alpha_3}(x^4 - 4\sqrt{3}x^3 + 15x^2 - 6\sqrt{3}x + 1)^{\alpha_4}. \quad (49)$$

In particular,

$$2\alpha_3 + 4\alpha_4 = s. \quad (50)$$

By (49), the coefficient of $P_1(x)$ by x^{s-2} is equal to

$$a_2 = \frac{3}{2}s^2 - \frac{5}{2}s + \alpha_4. \quad (51)$$

On the other hand, we have by Lemma 10

$$a_2 = f'_{11}(l'_1-2+l'_2\varepsilon^{-2})+f'_{12}(l'_1-1+(l'_2-1)\varepsilon^{-2})+f'_{21}((l'_1-1)\varepsilon^2+l'_2-1)+ \\ f'_{22}(l'_1\varepsilon^2+l'_2-2)+\frac{l'_1(l'_1-1)}{2}+\frac{l'_2(l'_2-1)}{2}+2l'_1l'_2, \quad (52)$$

where $f'_{11} = f_{11}$, $f'_{12} = f_{15} + f_{12}$, $f'_{21} = f_{51} + f_{21}$, $f'_{22} = f_{55} + f_{25}$, $l'_1 = l_1$, $l'_2 = l_5 + 1$. It follows from (52) that

$$a_2 = \frac{3}{2}s^2 - \frac{5}{2}s + f_{12} + f_{15}. \quad (53)$$

We obtain from (51), (53) that

$$f_{12} + f_{15} - \alpha_4 = 0. \quad (54)$$

Now, equations (45), (50), (54) implies that

$$f_{21} = 1 - \alpha_3 - \frac{1}{2}f_{15} - \frac{3}{2}f_{12}. \quad (55)$$

Since $f_{21} \geq 0$, it follows from (55) that there exist only three possibilities.

1. $a_3 = 1$, $f_{15} = f_{12} = 0$. Then $a_4 = 0$ and $s = 2$ which is a contradiction.

2. $a_3 = 0$, $f_{15} = f_{12} = 0$. Hence, $a_4 = 0$ and $s = 0$. This is a contradiction.

3. $a_3 = 0$, $f_{15} = 2$, $f_{12} = f_{21} = 0$, so that $a_4 = 2$ and $s = 8$. Direct computations show that there are no words $R(a, b)$ under our conditions such that $f_R(z) = \sqrt{3}z^8$. Thus Theorem 1 is proved in the case $l = 6$ and s is even.

2.3. The case $l > 6$

Let Γ be a group defined by (8). Taking into account Lemma 9, we can assume that 6 do not divide v_i for any i. Let us consider the epimorphic image Γ_1 of Γ:

$$\Gamma_1 = \langle c, d; c^2 = d^6 = R^2(c, d) = 1\rangle,$$

where $R(c, d) = cd^{v_1} \dots cd^{v_s}$. Since $6 \nmid v_i$ for any i, the word $R(c, d)$ from the free product $\langle c; c^2 = 1\rangle * \langle d; d^6 = 1\rangle$ can be written in the form $R(c, d) = cd^{u_1} \dots cd^{u_s}$ with $0 < u_i < 6$ and $u_i \equiv v_i \pmod 6$. We have already proved that Γ_1 contains a non-abelian free subgroup. Theorem 1 is proved.

3. Proof of Theorem 2

3.1. The case V is even.

Let us consider an epimorphism

$$\varphi : \Gamma \to \langle c; c^2 = 1\rangle, \quad \varphi(a) = 1, \varphi(b) = c.$$

Since $\varphi(R(a, b)) = 1$, we obtain using Reidemeister–Schreier rewriting process that $\ker\varphi$ has a representation of the form

$$\ker\varphi = \langle g_1, g_2, g_3; g_1^3 = g_2^3 = g_3^2 = R_1^2(g_1, g_2, g_3) = R_2^2(g_1, g_2, g_3) = 1\rangle,$$

where R_1 and R_2 is a rewriting of R. Let $F_3 = \langle g, h, t\rangle$ be a free group and $X(F_3)$ be the corresponding character variety. Consider a subvariety $W \subset X(F_3)$ defined by equations

$$\tau_g = \tau_h = 1, \quad \tau_t = \tau_{R_1(g,h,t)} = \tau_{R_2(g,h,t)} = 0.$$

It is easy to see that $W \neq \varnothing$. Indeed, by [1] for any generalized triangle group $T(n, m, l, R)$ there exists a special representation ρ of $T(n, m, l, R)$ into $\mathrm{PSL}_2(\mathbb{C})$, that is, a representation such that elements $\rho(a)$, $\rho(b)$ and $\rho(R)$ have orders n, m, l respectively. Let ρ be a special representation of Γ into $\mathrm{PSL}_2(\mathbb{C})$ and $\rho(g_1) = [A]$, $\rho(g_2) = [B]$, $\rho(g_3) = [C]$. We can choose matrices A, B such that $\operatorname{tr} A = \operatorname{tr} B = 1$. Then we shall have $\pi(A, B, C) \in W$, where π is defined by (3), so that $W \neq \varnothing$.

Let $W_1, \ldots, W_r$ be irreducible components of W. Since $\dim X(F_3) = 6$ and the subvariety $\varnothing \neq W \subset X(F_3)$ is defined by five equations, for any component W_i we must have $\dim W_i \geq 1$.

Lemma 13. $U_i = W_i \cap X^s(F_3) \neq \varnothing$.

Proof. Suppose that $U_i = \varnothing$ for some i. Then for any point $\rho = (A, B, C) \in \pi^{-1}(W_i)$ a group $\langle A, B, C\rangle$ is reducible. Without loss of generality we may assume that A, B, C are upper triangular matrices. Since A, B, C have finite orders, for any $S \in F_3$ the trace $\operatorname{tr} S(A, B, C) = \tau_S(\rho)$ can take only finite set of values, when $\rho \in \pi^{-1}(W_i)$. Hence, $\dim W_i = 0$ which is a contradiction. □

Let $\alpha_i : W_1 \to \mathbb{A}^1$ be a projection to the i-th coordinate. Since $\dim W_i \geq 1$, there exists i such that α_i is dominant. Let, for example, the projection α on the coordinate τ_{gh} is dominant, so that $\alpha(U_1)$ is dense in $\mathbb{A}^1$ in Zarisski topology. Hence, we can choose a transcendental number $\beta \in \mathbb{C}$ such that $\beta \in \alpha(U_1)$. Let $u \in \alpha^{-1}(\beta) \cap U_1$ and $(A, B, C) \in \pi^{-1}(u)$. By construction, we have $\operatorname{tr} A = \operatorname{tr} B = 1$, $\operatorname{tr} C = \operatorname{tr} R_1(A, B, C) = \operatorname{tr} R_2(A, B, C) = 0$.

Let $G = \langle [A], [B], [C] \rangle$. Let us show that G is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. First, G is irreducible by construction. Second, G is infinite since $\operatorname{tr} AB = \beta$ is a transcendental number, so that a matrix AB has infinite order. Third, G is not a dihedral group since $[A]$ has order 3.

Next, we have by construction

$$[A]^3 = [B]^3 = [C]^2 = R_1^2([A],[B],[C]) = R_2^2([A],[B],[C]) = 1.$$

Hence, G is an epimorphic image of $\ker\varphi$. Thus, $\ker\varphi$ contains a non-abelian free subgroup as required.

3.2. The case s is even.

Without loss of generality we can assume that V is odd. Set

$$f_R(z) = Q_R(1, \sqrt{2}, z),$$

where Q_R is the Fricke polynomial of the word $R = g^{u_1}h^{v_1}\dots g^{u_s}h^{v_s} \in F_2$. The leading coefficient of $F_R(z)$ is equal to

$$M_s = \prod_{i=1}^{s} P_{u_i-1}(1)P_{v_i-1}(\sqrt{2}) = (\sqrt{2})^t,$$

where t is a number of i such that $v_i = 2$.

Lemma 14. *Let us suppose that the polynomial $f_R(z)$ has a root $z_0 \notin \{0, \sqrt{2}, \frac{\sqrt{2}\pm\sqrt{6}}{2}\}$. Then Γ contains a non-abelian free subgroup.*

Lemma 14 can be proved in the same way as Lemma 8.

Bearing in mind Lemma 14, we may assume that the polynomial $f_R(z)$ has the form

$$f_R(z) = M_s z^{a_1}(z-\sqrt{2})^{a_2}(z-\frac{\sqrt{2}+\sqrt{6}}{2})^{a_3}(z-\frac{\sqrt{2}-\sqrt{6}}{2})^{a_4}. \quad (56)$$

Let ε be a primitive root of unity of degree 24, $F_2 = \langle g, h \rangle$ be a free group. Consider a representation $\rho : F_2 \to \mathrm{SL}_2(\mathbb{C})$ defined by

$$\rho(g) = A = \begin{pmatrix} \varepsilon^4 & 0 \\ 1 & \varepsilon^{-4} \end{pmatrix}, \qquad \rho(h) = B = \begin{pmatrix} \varepsilon^3 & x \\ 0 & \varepsilon^{-3} \end{pmatrix}.$$

Then $\operatorname{tr} A = 1$, $\operatorname{tr} B = \sqrt{2}$, $\operatorname{tr} AB = x + 2\cos(\frac{7\pi}{12}) = x - \frac{\sqrt{6}-\sqrt{2}}{2}$ and we have from (56)

$$f_1(x) = f_R(z)(\rho) = \operatorname{tr} R(A,B) = f_R(x - \frac{\sqrt{6}-\sqrt{2}}{2}) =$$
$$(\sqrt{2})^t(x-\frac{\sqrt{6}-\sqrt{2}}{2})^{a_1}(x-\frac{\sqrt{6}+\sqrt{2}}{2})^{a_2}(x-\sqrt{6})^{a_3}x^{a_4}. \quad (57)$$

The free coefficient of $\operatorname{tr} R(A,B)$ is equal to

$$\varepsilon^{4U+3V}+\varepsilon^{-4U-3V}=2\cos(\frac{4U+3V}{12}\pi), \tag{58}$$

where $U=\sum_{i=1}^{s}u_i$. Bearing in mind our assumptions, $2\cos(\frac{4U+3V}{12}\pi)$ can take only the following values:

$$\pm(\frac{\sqrt{6}-\sqrt{2}}{2})^{\pm 1}, \pm\sqrt{2}. \tag{59}$$

Then it follows from (57) that $a_4=0$.

Analogously, considering a representation $\rho_1 : F_2 \to \mathrm{SL}_2(\mathbb{C})$ defined by

$$\rho(g)=A=\begin{pmatrix}\varepsilon^4 & 0\\ 1 & \varepsilon^{-4}\end{pmatrix}, \qquad \rho(h)=B_1=\begin{pmatrix}\varepsilon^{-3} & x\\ 0 & \varepsilon^{3}\end{pmatrix},$$

we obtain $a_3=0$. Thus,

$$f_1(x)=(\sqrt{2})^t(x-\frac{\sqrt{6}-\sqrt{2}}{2})^{a_1}(x-\frac{\sqrt{6}+\sqrt{2}}{2})^{a_2}, \tag{60}$$

where $a_1+a_2=s$. Comparing constant terms of $f_1(x)$ and $\operatorname{tr} R(A,B_1)$, we obtain from (58), (60)

$$(\sqrt{2})^t(\frac{\sqrt{6}-\sqrt{2}}{2})^{a_1}(\frac{\sqrt{6}+\sqrt{2}}{2})^{a_2}=2\cos(\frac{4U+3V}{12}\pi). \tag{61}$$

Since $\frac{\sqrt{6}-\sqrt{2}}{2}\frac{\sqrt{6}+\sqrt{2}}{2}=1$ and s is even, it follows from (61) that $t=1$, $2a_1-s=0$, that is, $a_1=a_2=s/2$. Hence,

$$2\cos(\frac{4U+3V}{12}\pi)=\sqrt{2}.$$

Thus, we must have $U\equiv 0 \pmod 3$. But in this case there exists a well defined epimorphism

$$\lambda:\Gamma\to\langle d; d^3=1\rangle, \quad \lambda(a)=d, \lambda(b)=1.$$

Using Reidemeister–Schreier rewriting process, we obtain that $\ker\lambda$ has a representation of the form

$$\ker\lambda=\langle g_1,g_2,g_3; g_1^4=g_2^4=g_3^4= R_1^2(g_1,g_2,g_3)=R_2^2(g_1,g_2,g_3)=R_3^2(g_1,g_2,g_3)=1\rangle,$$

where R_1, R_2, R_3 are rewrites of R. One can check that $R_j(g_1,g_2,g_3)=g_{i_1}^{p_1}\dots g_{i_r}^{p_r}$, where $\sum_{i=1}^{r}p_i$ is even. By Theorem 1 from [3], $\ker\lambda$ (and consequently Γ) contains a non-abelian free subgroup. Theorem 2 is proved.

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