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# ON THE SOLUTION OF INTEGRAL EQUATIONS OF THE FIRST KIND WITH WEAK SINGULARITIES IN AN $n$-DIMENSIONAL SPHERICAL SHELL

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We study the integral equations

$$\int_{\Omega} |x-y|^{\alpha-n} \varphi(y)\,dy = f(x), \qquad x \in \Omega \subset \mathbf{R}^n,\ 0<\alpha<n, \tag{1}$$

$$\int_{\Omega} \ln|x-y| \varphi(y)\,dy = f(x), \qquad x \in \Omega \subset \mathbf{R}^n, \tag{2}$$

which are often encountered in the theory of function spaces with fractional smoothness [1]–[3], mathematical physics [4], and various applied problems (see, for example, [5]–[7]).

For $\Omega = \mathbf{R}^n$ the solutions of equations (1) and (2) are well known (see [2] and [3]). In the case of domains $\Omega$ different from the whole of $\mathbf{R}^n$ the investigation of solvability of these equations turns out to be more complicated in connection with the study of conditions on the right-hand sides $f$ on the boundary of $\Omega$. The case of a spherical domain $\Omega_b = \{x \in \mathbf{R}^n,\ |x| < b\}$ has been studied for equation (1). The solution of (1) for the disk $\Omega_1$ in $\mathbf{R}^2$ was first obtained in [7] in the form of a Fourier series with respect to an angular polar coordinate. An integrable solution of (1) under very stringent restrictions on $f$ and without solvability conditions was constructed in [8] and [9] for a disk and ball, respectively. Necessary and sufficient conditions for solvability of (1) in the space $L_p(\Omega_b; \rho) = \{f\colon \|\rho f; L_p(\Omega_b)\| < \infty\}$ with weight $\rho(x) = |x|^{\lambda}(b-|x|)^{\mu}$ were given in [10] and [11] in the radial ($f(x) = f(|x|)$) and nonradial cases. The solution of (2) was found in [12] and [13] for even and odd $n$ with respect to a spherical shell $\Omega_{a,b} = \{x \in \mathbf{R}^n\colon a < |x| < b\}$ and, in particular, for $a = 0$, with respect to the ball $\Omega_b$ in the radial case.

We obtain necessary and sufficient conditions for solvability of equations (1) and (2) with respect to an $n$-dimensional spherical shell, as well as explicit formulas for their solutions in an arbitrary nonradial case for even $\alpha$ with $0 < \alpha \le n$ in the space $C(\Omega)$ of continuous functions under the assumption that the free term $f$ belongs to a special space of differentiable functions. Writing the density $\varphi$ as a sum of two terms, of which one is the radial part of $\varphi$, we give integral representations for the Riesz potentials with respect to the spherical shell. They enable us to reduce (1) and (2) (for even $\alpha$ with $0 < \alpha \le n$) to the corresponding integral equations for the Fourier-Laplace coefficients $\varphi_{k,\mu}$ of the function $\varphi$, and both to find the solutions of the equations in the form of absolutely and uniformly convergent series in systems of spherical harmonics $Y_{k,\mu}$ and to find solvability conditions in the form of conditions on the Fourier-Laplace coefficients $f_{k,\mu}$ of the right-hand sides $f$.

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It is shown that the formulas for the solutions of (1) and (2) coincide with the power $(-\Delta)^{\alpha/2}$ of the Laplace operator $\Delta$.

We introduce the following notation: $\Omega_{a,b} = \{x \in \mathbf{R}^n : a < |x| < b,\ 0 \le a < b < \infty\}$ is a spherical shell in $\mathbf{R}^n$, $S_{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$ is the unit sphere in $\mathbf{R}^n$, $|S_{n-1}| = 2\pi^{n/2}\Gamma^{-1}(n/2)$ is the area of its surface, $P(x', \xi y') = |S_{n-1}|^{-1}(1-\xi^2) \times |x' - \xi y'|^{-n}$, $0 < \xi < 1$, $x, y \in S_{n-1}$, is the Poisson kernel for the unit ball;

$$c_{n,\alpha} = \begin{cases} 2^{-\alpha}\pi^{-n/2}\Gamma^{-1}(\alpha/2)\Gamma((n-\alpha)/2), & 0 < \alpha < n, \\ 2^{1-n}\pi^{-n/2}\Gamma^{-1}(n/2), & \alpha = n; \end{cases} \tag{3}$$

$I^\alpha_\Omega \varphi$ is the Riesz potential

$$I^\alpha_\Omega\varphi = \begin{cases} c_{n,\alpha}\int_\Omega |x-y|^{\alpha-n}\varphi(y)\,dy, & 0 < \alpha < n, \\ c_{n,n}\int_\Omega \ln|x-y|\varphi(y)\,dy, & \alpha = n; \end{cases} \tag{4}$$

$\gamma_{n,\alpha} = 2\Gamma^{-1}(\alpha)|S_{n-1}|^{-1}$, and $B^\alpha_{a+}\varphi$ and $B^\alpha_{b-}\varphi$ are the one-sided spherical potentials

$$B^\alpha_{a+}\varphi = \gamma_{n,\alpha}\int_{\Omega_{ax}} \frac{(|x|^2-|y|^2)^\alpha}{|x-y|^n}\varphi(y)\,dy, \qquad B^\alpha_{b-}\varphi = \gamma_{n,\alpha}\int_{\Omega_{xb}} \frac{(|y|^2-|x|^2)^\alpha}{|y-x|^n}\varphi(y)\,dy.$$

**Theorem 1.** *Suppose that $\varphi \in L_p(\Omega_{a,b};\rho)$, $\rho(x) = (|x|-a)^\lambda(b-|x|)^\mu$, $1 \le p \le \infty$, $\lambda < n/p'$ for $a = 0$ and $\lambda < 1/p'$ for $a > 0$, $\mu < 1/p'$ for $p > 1$, and $\lambda, \mu \le 0$ for $p = 1$. Let $\varphi_0(x)$ be the radial part of the function $\varphi$,*

$$\varphi_0(x) = |S_{n-1}|^{-1}\int_{S_{n-1}} \varphi(|x|x')\,d\sigma(x'),$$

*and let $\varphi_1 = \varphi - \varphi_0$. Then the Riesz potentials (4) have the representations*

$$I^\alpha_\Omega\varphi = I^\alpha_1\varphi + I^\alpha_2\varphi, \qquad 0 < \alpha < n, \tag{5}$$

$$I^n_\Omega\varphi = I^n_1\varphi_0 + I^n_2\varphi_1 + I^n_3\varphi_1, \tag{6}$$

*where*

$$I^\alpha_1\psi = 2^{-\alpha}B^{\alpha/2}_{a+}|y|^{-2}B^{\alpha/2}_{b-}\psi,$$

$$I^\alpha_2\psi = \left(\Gamma\left(\frac{\alpha}{2}\right)\right)^{-2}(2|x|)^{2-\alpha}\int_0^a s^{-1}(|x|^2-s^2)^{\alpha/2-1}\,ds$$
$$\times\int_a^b \rho(\rho^2-s^2)^{\alpha/2-1}\,d\rho\int_{S_{n-1}} P\left(x', \frac{s^2}{|x|\rho}y'\right)\psi(\rho y')\,d\sigma(y'),$$

$$I^n_3\psi = \frac{c_{n,n}}{|S_{n-2}|}\int_a^b \rho^{n-1}\psi(\rho)\,d\rho\int_{-1}^1 (1-t^2)^{(n-3)/2}\ln(|x|^2-2|x|\rho t+\rho^2)\,dt.$$

In the case of the ball $\Omega_b = \Omega_{0,b'}$ (5) coincides with the representation obtained in [11].

Following [14], we denote by $C^l(S_{n-1})$, $l = 1, 2, \dots$, the class of functions $f(x')$, $x' \in S_{n-1}$, such that $\delta^{\alpha/2}f \in C(S_{n-1})$, where $\delta$ is the Beltrami-Laplace operator. For odd $l$, $C^l(S_{n-1})$ is understood to be the closure of $C^\infty(S_{n-1})$ in the norm $\|f\|_C + \|\delta^{l/2}f\|_C$, where

$$\delta^{l/2}f = \sum_{k=0}^\infty [k(k+n-2)]^{l/2} f_{k,\mu}(|x|)Y_{k,\mu}(x'), \qquad f \in C^\infty(S_{n-1}),$$

$$f_{k,\mu}(|x|) = \int_{S_{n-1}} f(x)Y_{k,\mu}(x')\,d\sigma(x'), \qquad k = 0, 1, \dots, \mu = 1, 2, \dots,$$

$$d_n(k) = (n+k-2)(n+k-2)!/k!(n-2)!,$$

are the Fourier-Laplace coefficients of the function $f$ with respect to the system of spherical harmonics $Y_{k,\mu}(x')$ [2]. Denote by $C_l^N(\Omega_{ab})$ the space of bounded functions $f$ on $\Omega_{ab}$ such that the functions $\delta^j f(rx')$, $r = |x|$, $j = 0, 1, \ldots, l/2$, have continuous derivatives up to order $N$ with respect to the radial variable:

$$(\partial^k/\partial r^k)\delta^j f(rx') \in C(\Omega_{ab}), \qquad 1 \le k \le N;$$

$(\partial^N/\partial r^N) f(rx') \in C^l(S_{n-1})$ for any $r \in [a, b]$. Let $\Delta$ be the Laplace operator. We let

$$D = \frac{1}{r}\frac{d}{dr}, \qquad D_{\pm,k} = r\frac{d}{dr} \pm k, \qquad \Delta_k' = \frac{d}{dr}\Delta^k,$$

$$D_{m,n,k} = r^{2(m-k+1)-n} D^m r^{n+k-2},$$

$$(2m)!! = 2 \cdot 4 \cdots (2m), \qquad c_{n,n}' = (|S_{n-2}| c_{n,n})^{-1},$$

$$d_{ji} = \frac{(-1)^{m-j-1} \cdot 2^{j+i-m+1}\Gamma(n/2 + m + k - i - j - 1) a^{n+2(m+k-i-j-2)}}{\Gamma(m-i-1)\Gamma(m-j)\Gamma(n/2 - m + k)\Gamma(2m-i-j-1)},$$

$$d_k = \frac{(-1)^{m-k-1}[(m-1)!]^2}{(2m-2k-2)!k!},$$

$$\alpha(r, k) = b^{-2m+2k+2}\frac{d_k}{2(m-k+1)}(2m-2k-2)!!(4m-2k-4)!!r^{2m-1},$$

$$\beta(r, k, j) = \frac{(-1)^j r^{2(2m-k-j)-1}\alpha(r, k)}{(2m-2k-2j-4)!!(2m-2k-2j-6)!!},$$

$$\gamma(r, j, l) = \begin{cases} (-1)^l (2j)!! d_j a^{2m-2j-3} r^{2j-2l} & \text{if } a > 0, \\ 0 & \text{if } a < 0, \end{cases}$$

$$\gamma(r, i, j, l) = \frac{1}{2}(-1)^{j+l-1} c_{2m+2,2m} r^{2(2m-i-j-1)} \times \binom{m-1}{j} \frac{(2j)!!\Gamma(m-j)\Gamma(m-i)}{(2m-2j)!!(2j-2l)!!\Gamma(2m-i-j)}.$$

We have the following assertions.

**Theorem 2.** *Suppose that* $0 < d = 2m < n$, $m = 1, 2, \ldots$, $f \in C_l^N(\Omega_{ab})$, $N \ge 2m$, *and* $l > \frac{3}{2}n + 2m - 3$. *Equation* (1) *is solvable in* $C(\Omega_{a,b})$ *if and only if all the Fourier-Laplace coefficients* $f_{k,\mu}(|x|)$, $k = 0, 1, \ldots$, $\mu = 1, \ldots, d_n(k)$, *of* $f$ *satisfy the* $2m$ *conditions*

$$\sum_{j=0}^{m-1} d_{ji} D^j D_{m,n,k} f_{k,\mu}|_{|x|=a} = D^i |x|^{n+k-2} f_{k,\mu}|_{|x|=a}, \qquad i = 0, \ldots, m-1, \tag{7}$$

$$(D_{m,n,k} f_{k,\mu})^{(i)}|_{|x|=b} = 0, \qquad i = 0, 1, \ldots, m-1. \tag{8}$$

*Under these conditions the unique solution of* (1) *is given by*

$$\varphi(x) = (c_{n,\alpha})^{-1}(-1)^m \sum_{k=0}^{\infty}\sum_{\mu=1}^{d_n(k)} Y_{k,\mu}(x')|x|^k D^m D_{m,n,k} f_{k,\mu}(|x|), \tag{9}$$

*where the series converges absolutely and uniformly.*

**Theorem 3.** *Suppose that* $n = 2m$, $m = 1, 2, \dots$, $f \in C_l^N(\Omega_{ab})$, $N \geq n$, *and* $l > \frac{5}{2}n - 2$. *Equation* (3) *is solvable in* $C(\Omega_{ab})$ *if and only if the Fourier-Laplace coefficients* $f_{k,\mu}(|x|)$, $k = 1, 2, \dots, \mu = 1, \dots, d_n(k)$, *of* $f$ *satisfy conditions* (7) *and* (8), *and* $f_{0,1}(|x|)$ *satisfies the conditions*

$$\begin{aligned}
-|x|^{2m-2}(-\Delta)^{i+1} &+ \sum_{j=0}^{m-i-2}\sum_{i=0}^{j}\gamma(a, i, j, l)(-|x|^{2j-2l})D_{+,2m-2j+2l-2} \\
&\times \Delta^{m-l-1} f_{0,1}|_{|x|=a} \\
= \sum_{j=0}^{m-i-2}\sum_{l=0}^{j}&\gamma(b, i, j, l)|x|^{2j-2l}D_{+,2m-2j+2l-2} \\
&\times \Delta^{m-l-1} f_{0,1}|_{|x|=b}, \qquad i = 0, 1, \dots, m-2;
\end{aligned}$$

$$D^{m-i-1}(-\Delta)^{i+1} f_{0,1}|_{|x|=b} = 0, \qquad i = 0, 1, \dots, m-2,$$

$$\begin{aligned}
\lim_{|x|\to b}\Bigg((-1)^m c'_{2m,2m} &- |x|^{2m-1}\ln|x|\Delta'_{m-1} + \sum_{k=0}^{m-2}\Bigg[(-1)^{k-1}\alpha(|x|, k)\Delta'_k \\
&+ \sum_{j=0}^{m-k-1}(-1)^j\beta(|x|, k, j)D_{-,2m-2k-2j-4}\Delta^{m-j}\Bigg]\Bigg) f_{0,1}(|x|) \\
= \lim_{|x|\to a}\Bigg(-|x|^{2m-1}\ln|x|\Delta'_{m-1} &+ \sum_{k=0}^{m-2}\Bigg[(-1)^{k-1}\alpha(|x|, k)\Delta'_k \\
&+ \sum_{j=0}^{m-k-1}(-1)^j\beta(|x|, k, j)D_{-,2m-2k-2j-4}\Delta^{m-j}\Bigg]\Bigg) f_{0,1}(|x|),
\end{aligned}$$

$$\begin{aligned}
\lim_{|x|\to a}\Bigg[(-1)^m c'_{2m,2m}D &- \sum_{j=0}^{m-2}\sum_{l=0}^{j}\gamma(|x|, j, l)D_{+,2m-2j+2l-2}\Delta^{m-l-1}\Bigg] f_{0,1}(|x|) \\
&= \lim_{|x|\to b}\sum_{j=0}^{m-2}\sum_{l=0}^{j}\gamma(|x|, j, l)D_{+,2m-2j+2l-2}\Delta^{m-l-1} f_{0,1}(|x|).
\end{aligned}$$

*Under these conditions the unique solution of* (2) *is given by*

$$\begin{aligned}
\varphi(x) = (-1)^m(|S_{n-1}|)^{-1/2}D^{2m-1}&|x|^{2m-1}f_{0,1}(|x|) \\
+ \sum_{k=0}^{\infty}\sum_{\mu=1}^{d_n(k)}&|x|^k D^m|x|^{2-2k}D^m|x|^{2m+k-2}f_{k,\mu}(|x|),
\end{aligned} \tag{10}$$

*where the series converges absolutely and uniformly.*

**Lemma.** *If* $f \in C_l^N(\Omega_{ab})$, $N \geq 2m$, *and* $l > \frac{3}{2}n + 2m - 3$, *then the Fourier-Laplace coefficients* $f_{k,\mu}(|x|)$ *of* $f$ *satisfy*

$$Y_{k,\mu}(x')|x|^k D^m D_{m,n,k} f_{k,\mu}(|x|) = \Delta^m Y_{k,\mu}(x') f_{k,\mu}(|x|),$$
$$k = 1, 2, \dots, \quad \mu = 1, \dots, d_n(k).$$

**Theorem 4.** *Under the conditions of Theorems* 2 *and* 3 *the solutions* (9) *and* (10) *of equations* (1) *and* (2) *can be written in the form*

$$\varphi(x) = (c_{n,\alpha})^{-1}(-\Delta)^m f(x), \qquad 0 < \alpha = 2m \le n, \tag{11}$$

*where the* $c'_{n,\alpha}$ *are given by* (3).

In conclusion we note that (9)–(11) give explicit formulas for the inversion of the Riesz potentials (4) with respect to the $n$-dimensional spherical shell $\Omega = \Omega_{ab}$.

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