16D anisotropic inharmonic oscillator and 9D related (MICZ-)Kepler-like systems

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Abstract

We present some generalization of 16D oscillator by anisotropic and nonlinear inharmonic terms and its dual analog for 9D related MICZ-Kepler systems by generalized version of the Kustaanheimo-Stiefel transformation. The solvability of the Schrödinger equation of the these problems by the variables separation method are discussed in different coordinates.

1 Introduction

The oscillator and Kepler systems are the best known examples of exactly solvable tasks in few coordinate systems [1]. They are dual related with each other through the Hurwitz transformations in the dimensions of it's spaces, realizing the Hopf bundles. All favorite properties for given systems exist due to it's hidden symmetry. A few deformations of oscillator and Kepler systems can be realized by the appropriate reduction procedures of the initial hidden symmetry. There are anisotropic oscillator, nonlinear inharmonic oscillator, Kepler system with additional linear potential (its relevance to the Stark effect), two-center Kepler system [1] and so on [2]. The enumerous literature is devoted to this topic and has already developed academic logic and style of presentation of the materials for this topic (for example [3] - [8]).

On the other hand, in a number of their works, the authors [9] - [14] thoroughly worked out many questions of the last case of the Hopf bundle associated with 16-dimensional harmonic oscillator and 9-dimensional Coulomb problems. However, some moments turned out to be not considered by them and will be analyzed in this work. Thus, the goal of this paper is to give some generalization of 16D oscillator by anisotropic and nonlinear inharmonic terms and its dual analog for 9D related MICZ-Kepler systems by generalized version of the Kustaanheimo-Stiefel transformation.

The outline is as follows. As already mentioned first, we give the necessary information on result of the cycle of works by the authors of [9] - [14] in Section 2. Our main result will be the object of Sections 3 - 5 where the dual connection between the some deformations for the MICZ-Kepler system and 16-dimensional oscillator will be analyzed and shown. The exact analytical solutions of the Schrödinger equation for abovementioned problems are discused for a few coordinates systems. Brief concluding remarks will follow.

2 Dual connection between the 9D MICZ-Kepler and a 16D isotropic harmonic oscillator problems

Let us consider a 16 dimensional model, the Hamiltonian which is the sum of the two independent Hamiltonians of dimension 8, each with its own potential.

$$H = H_1 + H_2 = \sum_{1\leq a\leq 2} H_a = \sum_{1\leq a\leq 2}\left[-\frac{1}{2}\frac{\partial^2}{\partial x_s \partial x_s} + V(x_s x_s)\right] =$$

$$= \left[-\frac{1}{2}\frac{\partial^2}{\partial u_s \partial u_s} + V(u_s u_s)\right] + \left[-\frac{1}{2}\frac{\partial^2}{\partial v_s \partial v_s} + V(v_s v_s)\right], \qquad (1)$$

where u_s, v_t $(s,t = 1,\ldots,8)$ are Cartesian coordinates of the space $\mathbb{R}^{16}$. Here and futher on, 1) the small Greek letters λ, etc run from 1 to 9; 2) the small Latin letters j,k,s,t, etc run from 1 to 8; 3) use the Einstein convention: the repeated index is always summed up, if unless stated otherwise.

Further, we will keep in mind the independence of these 8D Hamiltonians from each other, which provide an anisotropic effect in our problem. If we choose a quadratic potential V_{ho} and one value of the oscillation frequency ω everywhere

$$V_{ho}(x_s x_s) = \frac{\omega_a^2 x_s x_s}{2}; \qquad (2)$$

$$\omega = \omega_1 = \omega_2, \qquad (3)$$

then we obtain the model of a 16-dimensional isotropic harmonic oscillator [9]

$$\hat{H}_0\psi(\mathbf{u},\mathbf{v}) = \left[-\frac{1}{2}\left(\frac{\partial^2}{\partial u_s \partial u_s} + \frac{\partial^2}{\partial v_s \partial v_s}\right) + \frac{\omega^2}{2}\left(u_s u_s + v_s v_s\right)\right]\psi = Z\psi, \qquad (4)$$

Here $\omega_a,\ Z_a$ $(Z = Z_1 + Z_2)$ have positive real values and are correspondingly the frequency and energy of the each harmonic oscillator.

According to [8], the Hurwitz transformation $\mathbb{R}^{16} \to \mathbb{R}^9$ that connects the 9D real space $\mathbb{R}^9$ of Cartesian coordinates $x_1, x_2, \ldots, x_9$ to the 16D real space $\mathbb{R}^{16}$ of Cartesian coordinates $u_1, u_2, \ldots, u_8, v_1, v_2, \ldots, v_8$ can be written as follows:

$$x_k \to 2(\Gamma_k)_{st} u_s v_t\ , \qquad x_9 \to u_s u_s - v_s v_s,, \qquad (5)$$

After the application of the Hurwitz transformation (5) to equation (4) the Schrödinger equation for the nine-dimensional MIC-Kepler problem in atomic units (thus $m = e = \hbar = 1$) has the form [12]

$$\hat{H}_0'\psi(\mathbf{r}) = \left(\frac{1}{2}\hat{\pi}^2 + \frac{\hat{Q}^2}{8r^2} - \frac{Z}{r}\right)\psi = E\psi, \tag{6}$$

in which Z is the nuclear charge of the Coulomb interaction, E is the energy of the particle, $\hat{\pi}^2 = \hat{\pi}_\lambda\hat{\pi}_\lambda$ with impulse operators are determined by

$$\hat{\pi}_j = -i\frac{\partial}{\partial x_j} + A_k\hat{Q}_{kj},\ \hat{\pi}_9 = -i\frac{\partial}{\partial x_9}, \tag{7}$$

where the potential vectors $A_k = \frac{x_k}{r(r+x_9)}$, squared operator $\hat{Q}^2 = \hat{Q}_{jk}\hat{Q}_{jk}$ and $r = \sqrt{x_\lambda x_\lambda} = u_s u_s + v_s v_s$ is distance in nine-dimensional space.

By plugging formulae (7) into Eq. (6) , the Schrödinger equation for $\hat{H}_0'$ now becomes according to works [13] - [14]

$$\hat{H}_0'\ \psi = \left[-\frac{\Delta}{2} + \frac{Q_{kj}L_{kj}}{2r(r+x_9)} + \frac{Q^2}{4r(r+x_9)} - \frac{Z}{r}\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{8}$$

We notify that in Eqs. (4) and (6) the roles of $E = -\frac{\omega^2}{2}$ and Z are interchanged. The variables E and Z become a negative number that denotes the energy of bound states and a parameter defining the 'charge' value in the Coulomb potential respectively.

Thus, in the paper [9] it is shown that a 16-dimensional isotropic harmonic oscillator and the nine-dimensional MICZ-Kepler problem described by the Schrödinger equation in Eqs. (4) and (6) are dual.

3 16D anisotropic and nonlinear inharmonic oscillator

In a work [15] we proposed next generalization of the Schrödinger equation for so-called a 16D double singular oscillator:

$$\hat{H}\ \psi(\mathbf{u},\mathbf{v}) = \left[\hat{H}_0 + \frac{c_1}{u_1^2 + ... + u_8^2} + \frac{c_2}{v_1^2 + ... + v_8^2}\right]\psi = Z\psi, \tag{9}$$

where c_1, c_2 nonnegative constants; $\hat{H}_0$ is Hamiltonian of a 16-dimensional isotropic harmonic oscillator determined earlier (4).

In other words, instead of harmonic oscillator potential V_{ho} in Eqs. (1), we chose the potential of a singular oscillator

$$V_{sho}(x_s x_s) = V_{ho}(x_s x_s) + \frac{c_a}{x_s x_s} = \frac{\omega^2 x_s x_s}{2} + \frac{c_a}{x_s x_s}. \tag{10}$$

To make variables separation in each 8D real space $\mathbb{R}^8$ where $x_1, x_2, \dots, x_8$ are Cartesian coordinates, we have introduced it's hyperspherical coordinates: $(\phi_1, \dots, \phi_7)$ are the hyperspherical angles and r is the hyperradius by

$$\begin{aligned} x_8 &= r\cos(\phi_7), \\ x_7 &= r\sin(\phi_7)\cos(\phi_6), \\ &\dots \\ &\dots \\ x_2 &= r\sin(\phi_7)\sin(\phi_6)\cdots\sin(\phi_2)\cos(\phi_1), \\ x_1 &= r\sin(\phi_7)\sin(\phi_6)\cdots\sin(\phi_2)\sin(\phi_1), \end{aligned} \tag{11}$$

Using the ansatz $\Psi(\mathbf{r}) = \Psi(r, \phi_1, \cdots, \phi_7) = R(r)\Omega(\phi)$ the Schrödinger equation for each H_a can be rewritten in terms of it's separated equations of hyperradius r and angular variables ϕ as follows:

$$\left[\frac{1}{r^7}\frac{\partial}{\partial r}(r^7\frac{\partial}{\partial r}) + \left(2Z_a - \omega^2 r^2\right) - \frac{\Lambda^2 + 2c_a}{r^2}\right] R(r) = 0, \tag{12}$$

$$\left[\Lambda^2 - L(L+6)\right]\Omega(\phi) = 0, \tag{13}$$

where Z_a are the eigenvalues of H_a, but $Z_1 + Z_2 = Z$; $\Lambda = L(L+6)$ is the separation constant and is also an eigenvalue of the operator Λ^2 (13).

A solution of Eq. (12) are given in terms of a special function

$$R_{NL}(r) = C_{NL} r^{L'} e^{-\frac{\omega r^2}{2}} {}_1F_1(-N; L'+4; \omega r^2) \tag{14}$$

where $L'(L'+6) = L(L+6) + 2c_a$.

Thus, it can be said that, firstly, the actual multidimensional problem was reduced to one-dimensional case for hyperradius r, and, secondly, our proposed generalization according to the above formulas is reduced to additive terms in the negative power of r for the initial Hamiltonians.

On the other hand, recall, that there are quasi-exactly problems which occupy an intermediate place between exactly solvable problems and non-solvable ones. The theory of quasi-exactly systems (QES) gives the next generalization or the family of potentials in the direction of degrees r less than 2 for a fixed N, by

$$V_{<2}(r) \; = V_{sho} + ar + \frac{b}{r} \; = \; \frac{\omega^2 r^2}{2} + \frac{c}{r^2} + ar + \frac{b}{r} \tag{15}$$

with the eigenfunctions

$$R(r) \; = \; p_{N-1}(r) r^{l'-c'} e^{-\frac{b'}{2}r^2 - a' r} \;, \tag{16}$$

where there are the following reassignment of constants from work [16] to our designation $\omega^2 = 2b'^2$; $a = 2a'b'$; $b = -a'(D-2c')$; $c = c'(c'-D+1)$; $d = a'^2 - b'(2N+D-1-2c')$; $D = d' + 2l' - 1$; $p_{N-1}(r)$ are polynomial of the $(N-1)$-th degree.

This QES potential appears in a number of applications to the systems with two electrons ([17] - [18]).

Also at present we want to go to oppositive direction and to consider the potential which depenfing on u, v for it's degrees more than 2. The QES theory gives the other generalization in this direction or the family of potentials for a fixed N, by see [16],

$$V_{>2}(z) = V_{sho} + br^4 + ar^6 \ = (\omega r)^2/2 + \frac{c}{r^2} + br^4 + ar^6 \ , \qquad (17)$$

with the eigenfunctions

$$R(r) \ = \ p_{N-1}(r^2) r^{l'-c'} e^{-\frac{a' r^4}{4} - \frac{b' r^2}{2}}, \qquad (18)$$

where the following reassignment of constants from work [16] to our designation $\omega^2 = 2[b'^2 - (4N + D - 2c' - 1)a']; \quad a = a'^2; \quad b = 2a'b'; \quad c = c'(c' - D + 1)$.

The one-dimensional Hamiltonian of the nonrelativistic quantum systems with this anharmonic potential (17) is well known as the crucial example that stimulated the investigation of quasi-exactly solvable systems.

Thus, we offer four different models of the 16D anisotropic and nonlinear anharmonic oscillator in QES class. Each model is represented by a sum of two independent oscillators of dimensions $D = 8$ with various anisotropic and nonlinear anharmonic terms of the potential. In other words, we will further consider the following Hamiltonians of dimension $D = 8$:

$$\begin{aligned} {}_{<2}H = & \left[-\tfrac{1}{2}\tfrac{\partial^2}{\partial x_s \partial x_s} + V_{<2}(x_s x_s)\right]; \\ V_{<2}(x_s x_s) = & V_{sho}(x_s x_s) + \tfrac{b}{\sqrt{x_s x_s}} + a\sqrt{x_s x_s} \\ = & \tfrac{\omega^2 x_s x_s}{2} + \tfrac{c}{x_s x_s} + \tfrac{b}{\sqrt{x_s x_s}} + a\sqrt{x_s x_s} \end{aligned} \qquad (19)$$

and

$$\begin{aligned} {}_{>2}H = & \left[-\tfrac{1}{2}\tfrac{\partial^2}{\partial x_s \partial x_s} + V_{>2}(x_s x_s)\right]; \\ V_{>2}(x_s x_s) = & V_{sho}(x_s x_s) + b(x_s x_s)^2 + a(x_s x_s)^3 \\ = & \tfrac{\omega^2 x_s x_s}{2} + \tfrac{c}{x_s x_s} + b(x_s x_s)^2 + a(x_s x_s)^3. \end{aligned} \qquad (20)$$

In hyperspherical coordinates, the potentials of these Hamiltonians are successively reduced to the potential (15) and (17). In any case, the final wave function $\psi(\mathbf{u}, \mathbf{v})$ of Eq. (1) will be represented by the product of the wave functions of each oscillator $\Psi(\mathbf{r_a}) \equiv R(r_a)\Omega(\phi_a)$ of Eq. (19) - (20):

- Model 1

$$\begin{aligned} H\psi_1(\mathbf{u}, \mathbf{v}) &= [{}_{<2}H_1 + {}_{<2}H_2]\, \psi_{<2}(\mathbf{u})\psi_{<2}(\mathbf{v}) \\ &\equiv [{}_{<2}H_1 + {}_{<2}H_2]\, \Psi_{<2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2}) \\ &= {}_{<2}H_1 \left[\Psi_{<2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2})\right] +_{<2} H_2 \left[\Psi_{<2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2})\right] \end{aligned}$$

$$
\begin{aligned}
&= \Psi_{<2}(\mathbf{r_2})\left[{}_{<2}H_1\Psi_{<2}(\mathbf{r_1})\right] + \Psi_{<2}(\mathbf{r_1})\left[{}_{<2}H_2\Psi_{<2}(\mathbf{r_2})\right] \\
&= \Psi_{<2}(\mathbf{r_2})\left[Z_1\Psi_{<2}(\mathbf{r_1})\right] + \Psi_{<2}(\mathbf{r_1})\left[Z_2\Psi_{<2}(\mathbf{r_2})\right] \\
&= \left[Z_1 + Z_2\right]\Psi_{<2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2}) \equiv Z\Psi_{<2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2}) \\
&\equiv ZR_{<2}(r_1)\Omega_1(\phi_1)R_{<2}(r_2)\Omega_2(\phi_2) = Z\psi_1(\mathbf{u},\mathbf{v}) \qquad (21)
\end{aligned}
$$

- Model 2

$$
\begin{aligned}
H\psi_2(\mathbf{u},\mathbf{v}) &= \left[{}_{<2}H_1 + {}_{>2}H_2\right]\psi_{<2}(\mathbf{u})\psi_{>2}(\mathbf{v}) \\
&\equiv Z\Psi_{<2}(\mathbf{r_1})\Psi_{>2}(\mathbf{r_2}) \qquad (22)
\end{aligned}
$$

- Model 3

$$
\begin{aligned}
H\psi_3(\mathbf{u},\mathbf{v}) &= \left[{}_{>2}H_1 + {}_{<2}H_2\right]\psi_{>2}(\mathbf{u})\psi_{<2}(\mathbf{v}) \\
&\equiv Z\Psi_{>2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2}) \qquad (23)
\end{aligned}
$$

- Model 4

$$
\begin{aligned}
H\psi_4(\mathbf{u},\mathbf{v}) &= \left[{}_{>2}H_1 + {}_{>2}H_2\right]\psi_{>2}(\mathbf{u})\psi_{>2}(\mathbf{v}) \\
&\equiv Z\Psi_{>2}(\mathbf{r_1})\Psi_{>2}(\mathbf{r_2}) \qquad (24)
\end{aligned}
$$

Recall that the hyperradius part $R(r_a)$ of the eigenfunction $\Psi(\mathbf{r_a})$ has the form (16) and (18), respectively, depending on the potential (15) and (17).

4 9D related (MICZ-) Kepler-like systems as dual analog of 16D anisotropic and nonlinear inharmonic oscillator

The Hurwitz transformation links the harmonic oscillator with the Coulomb problem. Therefore, 9D related MICZ-Kepler systems are considered as dual analog of 16D oscillator. After the application of the Hurwitz transformation (5) to equation (4) and by plugging formulae (7) into Eq. (6), the Schrödinger equation for the nine-dimensional MICZ-Kepler problem in atomic units (thus $m = e = \hbar = 1$) $\hat{H}_0'$ now becomes according to works [13] - [14]

$$
\hat{H}_0' \; \psi = \left[-\frac{\Delta}{2} + \frac{J^2 - L^2}{4r(r + x_9)} + V_C'\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}). \qquad (25)
$$

where $E = -\frac{\omega^2}{2}, V_C' = -\frac{Z}{r}$ and $J_{kj} = L_{kj} + Q_{kj}$.

This means that this transition from one equation (4) to another (6-8) or (25) can be interpreted as some change of expression

$$
\sum_{1 \le a \le 2} \left[V_{iho}(x_s x_s) - Z\right] \quad \equiv V_{iho1}(u_s u_s) + V_{iho2}(v_s v_s) - Z_1 - Z_2 \quad =
$$

$$\sum_{1\le a\le 2}\left[\omega^2 x_s x_s/2 - Z\right] \equiv \omega^2 u_s u_s/2 - Z_1 + \omega^2 v_s v_s/2 - Z_2 =$$

$$\sum_{1\le a\le 2}\left[-E x_s x_s - Z\right] \equiv -E_1 u_s u_s - Z_1 - E_2 v_s v_s - Z_2$$

$$\rightarrow V'_{JLC} - E \equiv V'_{JL} + V'_C - E \equiv \frac{J^2-L^2}{4r(r+x_9)} - \frac{Z}{r} - E$$

or

$$\sum_{1\le a\le 2}\left[V_{iho}(x_s x_s) - Z\right] = \sum_{1\le a\le 2}\left[\omega^2 x_s x_s/2 - Z\right] \equiv \sum_{1\le a\le 2}\left[-E x_s x_s - Z\right]$$

$$\rightarrow V'_{JLC} - E = \frac{1}{r}\left[\frac{J^2-L^2}{4(r+x_9)} - Z - Er\right] \equiv \frac{1}{r}\left[rV'_{JL} + r(V'_C - E)\right]$$

Considering the additive and singular terms $c_a/x_s x_s$ to the potential of a harmonic oscillator $V_{ho}(x_s x_s)$, we obtain the potential of a singular harmonic oscillator $V_{sho}(x_s x_s)$ (10), which after Hurwitz transformation will also have an additional additive terms to the previously obtained Coulomb potential V'_C. Therefore, we get the following $9D$ generalized MIC-Kepler system with non central terms

$$\hat{H}'\ \psi(\mathbf{r}) = \left[\hat{H}'_0 + \frac{2c_1}{r(r+x_9)} + \frac{2c_2}{r(r-x_9)}\right]\psi = E\psi, \tag{26}$$

where $\hat{H}'_0$ is Hamiltonian of the nine-dimensional MICZ-Kepler problem determined earlier (25).

This was not hard to do if is remember what is taking place:

$$r = \sqrt{x_\lambda x_\lambda} = u_s u_s + v_s v_s; \qquad x_9 = u_s u_s - v_s v_s$$
$$2u_s u_s = r + x_9; \qquad 2v_s v_s = r - x_9.$$

Thus, now we are ready to determine the potential of the dual analog for our generalizations of the 16D oscillator [4 models (21) - (24] as 9D related MIC-Kepler systems in different coordinates. The solvability of the Schrödinger equation of the these problems by the variables separation method will be discussed in spherical and parabolic coordinates.

4.1 Variables separation in spherical coordinates

According to to works [13] - [14], the Cartesian coordinates $x_1, x_2, \ldots, x_9$ of 9D real space $\mathbb{R}^9$ are defined by the nine-dimensional spherical coordinates

$$\begin{aligned}
x_9 &= r\cos(\theta),\\
x_8 &= r\sin(\theta)\cos(\phi_6),\\
&\ldots\\
&\ldots\\
x_2 &= r\sin(\theta)\sin(\phi_6)\cdots\sin(\phi_1)\cos(\phi_0),\\
x_1 &= r\sin(\theta)\sin(\phi_6)\cdots\sin(\phi_1)\sin(\phi_0),
\end{aligned} \tag{27}$$

Given these definitions, we obtain useful relations:

$$2u_s u_s = r + x_9 \equiv r(1+\cos(\theta)) = 2r\cos^2(\frac{\theta}{2});$$
$$2v_s v_s = r - x_9 \equiv r(1-\cos(\theta)) = 2r\sin^2(\frac{\theta}{2}).$$

In other words, we received next typies of potential in spherical coordinates

$$\begin{aligned}
V_{<2}(u_s u_s) &= \frac{\omega_1^2 u_s u_s}{2} + \frac{c_1}{u_s u_s} + \frac{b_1}{\sqrt{u_s u_s}} + a_1\sqrt{u_s u_s} \\
&= \frac{\omega_1^2 r\cos^2(\frac{\theta}{2})}{2} + \frac{c_1}{r\cos^2(\frac{\theta}{2})} + \frac{b_1}{\sqrt{r\cos^2(\frac{\theta}{2}))}} + a_1\sqrt{r\cos^2(\frac{\theta}{2})} \\
&= r\left[-E_1\cos^2(\frac{\theta}{2}) + \frac{c_1}{r^2\cos^2(\frac{\theta}{2})} + \frac{b_1}{\sqrt{r^3\cos^2(\frac{\theta}{2})}} + a_1\sqrt{\frac{\cos^2(\frac{\theta}{2})}{r}}\right] \\
V_{<2}(v_s v_s) &= r\left[-E_2\sin^2(\frac{\theta}{2}) + \frac{c_2}{r^2\sin^2(\frac{\theta}{2})} + \frac{b_2}{\sqrt{r^3\sin^2(\frac{\theta}{2})}} + a_2\sqrt{\frac{\sin^2(\frac{\theta}{2})}{r}}\right]
\end{aligned}$$

and

$$\begin{aligned}
V_{>2}(u_s u_s) &= \frac{\omega_1^2 u_s u_s}{2} + \frac{c_1}{u_s u_s} + b_1(u_s u_s)^2 + a_1(u_s u_s)^3 \\
&= \frac{\omega_1^2 r\cos^2(\frac{\theta}{2})}{2} + \frac{c_1}{r\cos^2(\frac{\theta}{2})} + b_1(r\cos^2(\frac{\theta}{2}))^2 + a_1(r\cos^2(\frac{\theta}{2}))^3 \\
&= r\left[-E_1\cos^2(\frac{\theta}{2}) + \frac{c_1}{r^2\cos^2(\frac{\theta}{2})} + b_1 r\cos^4(\frac{\theta}{2}) + a_1 r^2\cos^6(\frac{\theta}{2})\right] \\
V_{>2}(v_s v_s) &= r\left[-E_2\sin^2(\frac{\theta}{2}) + \frac{c_2}{r^2\sin^2(\frac{\theta}{2})} + b_2 r\sin^4(\frac{\theta}{2}) + a_2 r^2\sin^6(\frac{\theta}{2})\right]
\end{aligned}$$

This means that all additive terms to the Coulomb potential have a double dependence both on the hyperradius r and on the angular variable θ. It is reasonable to assume that: 1) their separability is possible only when the entire dependence on the angular variable θ is collected with one functional dependence on the hyperradius r, and in other cases it is absent; 2) the type of functional dependence on hyperradius r is determined by the existing functional dependence on hyperradius r near derivatives with respect to the angular variable $\frac{\partial}{\partial\theta}$.

Following this logic leads to the scheme of separation of variables for the (generalized) Coulomb case with the amendment in the equation for the radial variable, where additional terms appear without angular dependence, i.e. instead of $-r(V_C' - E) = Z + Er$ we get $W'(r)$. Thus, the Schrödinger equation of our generalized MICZ-Kepler system with new terms rewritten in terms of next separated equations of variables r, θ and angular variables ϕ, ϕ'

$$\left[\frac{1}{r^8}\frac{\partial}{\partial r}(r^8\frac{\partial}{\partial r}) - \frac{2}{r}W'(r) - \frac{\Lambda}{r^2}\right] R(r) = 0, \quad (28)$$

$$\left[\frac{1}{\sin^7\theta}\frac{\partial}{\partial\theta}(\sin^7\theta\frac{\partial}{\partial\theta}) - \frac{L(L+6)+8c_2}{4\sin^2\frac{\theta}{2}} - \frac{J(J+6)+8c_1}{4\cos^2\frac{\theta}{2}} + \Lambda\right]\Theta(\theta) = 0, \quad (29)$$

$$\left[J^2 - J(J+6)\right]\Phi(\phi,\phi') = 0, \quad (30)$$

$$\left[L^2 - L(L+6)\right]\Phi(\phi,\phi') = 0, \quad (31)$$

where $\Lambda = \lambda(\lambda+7)$ is the separation constant and is also an eigenvalue of the operator (29).

Let us specify the value of $W'(r) \equiv \frac{1}{r}\left[V(u_s u_s) + V(v_s v_s) - Z_1 - Z_2\right]$ for each model and the conditions on coefficients of potential for (quasi) exact solvability according to above logic.

- Model 1

$$\begin{aligned} H\psi_1(\mathbf{u},\mathbf{v}) = & \left[{}_{<2}H_1 + {}_{<2}H_2\right]\psi_{<2}(\mathbf{u})\psi_{<2}(\mathbf{v}) \equiv Z\psi_1(\mathbf{u},\mathbf{v}) \\ W'(r) = & \left[-E_1\cos^2(\tfrac{\theta}{2}) + \tfrac{b_1}{\sqrt{r^3\cos^2(\frac{\theta}{2})}} + a_1\sqrt{\tfrac{\cos^2(\frac{\theta}{2})}{r}}\right] - Z_1 \\ + & \left[-E_2\sin^2(\tfrac{\theta}{2}) + \tfrac{b_2}{\sqrt{r^3\sin^2(\frac{\theta}{2})}} + a_2\sqrt{\tfrac{\sin^2(\frac{\theta}{2})}{r}}\right] - Z_2 \end{aligned}$$

  or $E_1 = E_2;\; b_1 = a_1 = b_2 = a_2 = 0.$

- Model 2

$$\begin{aligned} H\psi_2(\mathbf{u},\mathbf{v}) = & \left[{}_{<2}H_1 + {}_{>2}H_2\right]\psi_{<2}(\mathbf{u})\psi_{>2}(\mathbf{v}) \equiv Z\Psi_{<2}(\mathbf{r_1})\Psi_{>2}(\mathbf{r_2}) \\ W'(r) = & \left[-E_1\cos^2(\tfrac{\theta}{2}) + \tfrac{b_1}{\sqrt{r^3\cos^2(\frac{\theta}{2})}} + a_1\sqrt{\tfrac{\cos^2(\frac{\theta}{2})}{r}}\right] - Z_1 \\ + & \left[-E_2\sin^2(\tfrac{\theta}{2}) + b_2 r\sin^4(\tfrac{\theta}{2}) + a_2 r^2\sin^6(\tfrac{\theta}{2})\right] - Z_2 \end{aligned}$$

  or $E_1 = E_2;\; b_1 = a_1 = b_2 = a_2 = 0.$

- Model 3

$$\begin{aligned} H\psi_3(\mathbf{u},\mathbf{v}) = & \left[{}_{>2}H_1 + {}_{<2}H_2\right]\psi_{>2}(\mathbf{u})\psi_{<2}(\mathbf{v}) \equiv Z\Psi_{>2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2}) \\ W'(r) = & \left[-E_1\cos^2(\tfrac{\theta}{2}) + b_1 r\cos^4(\tfrac{\theta}{2}) + a_1 r^2\cos^6(\tfrac{\theta}{2})\right] - Z_1 \\ + & \left[-E_2\sin^2(\tfrac{\theta}{2}) + \tfrac{b_2}{\sqrt{r^3\sin^2(\frac{\theta}{2})}} + a_2\sqrt{\tfrac{\sin^2(\frac{\theta}{2})}{r}}\right] - Z_2 \end{aligned}$$

  or $E_1 = E_2;\; b_1 = a_1 = b_2 = a_2 = 0.$

- Model 4

$$\begin{array}{rl} H\psi_4(\mathbf{u},\mathbf{v}) & = [{}_{>2}H_1 + {}_{>2}H_2]\,\psi_{>2}(\mathbf{u})\psi_{>2}(\mathbf{v}) \equiv Z\Psi_{>2}(\mathbf{r_1})\Psi_{>2}(\mathbf{r_2}) \\ W'(r) = & \left[-E_1\cos^2(\frac{\theta}{2}) + b_1 r\cos^4(\frac{\theta}{2}) + a_1 r^2\cos^6(\frac{\theta}{2})\right] - Z_1 \\ + & \left[-E_2\sin^2(\frac{\theta}{2}) + b_2 r\sin^4(\frac{\theta}{2}) + a_2 r^2\sin^6(\frac{\theta}{2})\right] - Z_2 \end{array}$$

or $E_1 = E_2;\ b_1 = a_1 = b_2 = a_2 = 0.$

Thus, we have obtained that in spherical coordinates the maximum (quasi) exactly solvable model for our proposal is only the (generalized) Coulomb model (26).

4.2 Variables separation in parabolic coordinates

According to works [13] - [14], the Cartesian coordinates $x_1, x_2, \ldots, x_9$ of 9D real space $\mathbb{R}^9$ are defined by the parabolic coordinates on the S_7 sphere

$$\begin{aligned} x_9 &= \frac{u-v}{2}, \\ x_8 &= \sqrt{uv}\cos(\phi_6), \\ &\ldots \\ &\ldots \\ x_2 &= \sqrt{uv}\sin(\phi_6)\cdots\cos(\phi_0), \\ x_1 &= \sqrt{uv}\sin(\phi_6)\cdots\sin(\phi_0), \\ r &= \frac{u+v}{2}. \end{aligned} \tag{32}$$

The Schrodinger equation $H'\Psi = E\Psi$ (25) is same, but the Laplace-Beltrami operator Δ in the hyperparabolic coordinates has the next form:

$$\Delta = \frac{4}{u+v}\left\{\frac{1}{u^3}\frac{\partial}{\partial u}(u^4\frac{\partial}{\partial u}) + \frac{1}{v^3}\frac{\partial}{\partial r}(v^4\frac{\partial}{\partial v})\right\} - \frac{L^2}{uv}$$

Given these definitions, we obtain useful relations:

$$\begin{array}{ll} r = \sqrt{x_\lambda x_\lambda} = u_s u_s + v_s v_s; & x_9 = u_s u_s - v_s v_s \\ 2u_s u_s = r + x_9 \equiv u; & 2v_s v_s = r - x_9 \equiv v \end{array}$$

In other words, we received next typies of potential in the parabolic coordinates

$$\begin{aligned} V_{<2}(u_s u_s) &= \frac{\omega_1^2 u_s u_s}{2} + \frac{c_1}{u_s u_s} + \frac{b_1}{\sqrt{u_s u_s}} + a_1\sqrt{u_s u_s} \\ &= \frac{-uE_1}{2} + \frac{2c_1}{u} + b_1\sqrt{\frac{2}{u}} + a_1\sqrt{\frac{u}{2}} \\ V_{<2}(v_s v_s) &= \frac{-vE_2}{2} + \frac{2c_2}{v} + b_2\sqrt{\frac{2}{v}} + a_2\sqrt{\frac{v}{2}} \end{aligned}$$

and

$$
\begin{aligned}
V_{>2}(u_s u_s) &= \frac{\omega_1^2 u_s u_s}{2} + \frac{c_1}{u_s u_s} + b_1 (u_s u_s)^2 + a_1 (u_s x_s)^3 \\
&= \frac{-uE_1}{2} + \frac{2c_1}{u} + \frac{b_1 u^2}{4} + \frac{a_1 u^3}{8} \\
V_{>2}(v_s v_s) &= \frac{-uE_2}{2} + \frac{2c_2}{u} + \frac{b_2 u^2}{4} + \frac{a_2 u^3}{8}
\end{aligned}
$$

As in the case of spherical coordinates, we have additional additive terms, but each of which now depends on only one variable. This fact makes it easy to separate variables according to the scheme of separation of variables for the (generalized) Coulomb case.

Thus, the Schrödinger equation of our generalized MIC-Kepler system with new terms in the hyperparabolic coordinates can be rewritten in terms of separated equations of variables u, v and angular variables ϕ, ϕ'

$$\left[\frac{1}{u^3}\frac{\partial}{\partial u}(u^4 \frac{\partial}{\partial u}) - \frac{J(J+6)+8c_1}{4u} - W'_u(r) - P\right] U(u) = 0, \tag{33}$$

$$\left[\frac{1}{v^3}\frac{\partial}{\partial v}(v^4 \frac{\partial}{\partial v}) - \frac{L(L+6)+8c_2}{4v} - W'_v(r) + P\right] V(v) = 0, \tag{34}$$

$$\left[J^2 - J(J+6)\right] \Phi(\phi, \phi') = 0, \tag{35}$$

$$\left[L^2 - L(L+6)\right] \Phi(\phi, \phi') = 0, \tag{36}$$

where P is the separation constant.

Let us specify the value of $W'_u(r) \equiv [V(u_s u_s) - Z_1]$ and $W'_v(r) \equiv [V(v_s v_s) - Z_2]$ for each model

- Model 1

$$
\begin{aligned}
H\psi_1(\mathbf{u}, \mathbf{v}) &= \left[{}_{<2}H_1 + {}_{<2}H_2\right] \psi_{<2}(\mathbf{u})\psi_{<2}(\mathbf{v}) \equiv Z\psi_1(\mathbf{u}, \mathbf{v}) \qquad (37) \\
W'_u(r) &= \left[\tfrac{-uE_1}{2} + b_1\sqrt{\tfrac{2}{u}} + a_1\sqrt{\tfrac{u}{2}}\right] - Z_1 \\
W'_v(r) &= \left[\tfrac{-vE_2}{2} + b_2\sqrt{\tfrac{2}{v}} + a_2\sqrt{\tfrac{v}{2}}\right] - Z_2
\end{aligned}
$$

- Model 2

$$
\begin{aligned}
H\psi_2(\mathbf{u}, \mathbf{v}) &= \left[{}_{<2}H_1 + {}_{>2}H_2\right] \psi_{<2}(\mathbf{u})\psi_{>2}(\mathbf{v}) \equiv Z\Psi_{<2}(\mathbf{r_1})\Psi_{>2}(\mathbf{r_2}) \qquad (38) \\
W'_u(r) &= \left[\tfrac{-uE_1}{2} + b_1\sqrt{\tfrac{2}{u}} + a_1\sqrt{\tfrac{u}{2}}\right] - Z_1 \\
W'_v(r) &= \left[\tfrac{-uE_2}{2} + \tfrac{b_2 u^2}{4} + \tfrac{a_2 u^3}{8}\right] - Z_2
\end{aligned}
$$

- Model 3

$$
\begin{aligned}
H\psi_3(\mathbf{u}, \mathbf{v}) &= \left[{}_{>2}H_1 + {}_{<2}H_2\right] \psi_{>2}(\mathbf{u})\psi_{<2}(\mathbf{v}) \equiv Z\Psi_{>2}(\mathbf{r_1})\Psi_{<2}(\mathbf{r_2}) \qquad (39) \\
W'_u(r) &= \left[\tfrac{-uE_1}{2} + \tfrac{b_1 u^2}{4} + \tfrac{a_1 u^3}{8}\right] - Z_1 \\
W'_v(r) &= \left[\tfrac{-vE_2}{2} + b_2\sqrt{\tfrac{2}{v}} + a_2\sqrt{\tfrac{v}{2}}\right] - Z_2
\end{aligned}
$$

- Model 4

$$
\begin{aligned}
H\psi_4(\mathbf{u},\mathbf{v}) &= \left[{}_{>2}H_1 + {}_{>2}H_2\right]\psi_{>2}(\mathbf{u})\psi_{>2}(\mathbf{v}) \equiv Z\Psi_{>2}(\mathbf{r_1})\Psi_{>2}(\mathbf{r_2}) \quad (40)\\
W'_u(r) = & \quad \left[\tfrac{-uE_1}{2} + \tfrac{b_1u^2}{4} + \tfrac{a_1u^3}{8}\right] - Z_1\\
W'_v(r) = & \quad \left[\tfrac{-uE_2}{2} + \tfrac{b_2u^2}{4} + \tfrac{a_2u^3}{8}\right] - Z_2
\end{aligned}
$$

At first glance, it seems that 4 models require solving 2 qualitatively different types of problems. However, getting rid of irrationality by introducing a new variable $x' \to x^2$ in one type of problem leads to the solution of a second type problem, which in turn coincides with the problem we considered earlier (17) with solution (18).

Thus, we have obtained that in parabolic coordinates all proposed 4 models (37) - (40) is the (quasi) exactly solvable models.

Now let us compare the results obtained with those available in the literature [6]. If we consider just a problem with potential (2), but without a condition (3), then we obtain in the 16D the sum of two independent harmonic oscillators that will be dual to the 9D MICZ-problem with the potential $\cos\theta$:

$$
\begin{aligned}
\frac{\omega_1^2 u_s u_s}{2} + \frac{\omega_2^2 v_s v_s}{2} & \equiv -E_1 u_s u_s - E_2 v_s v_s\\
\to \frac{1}{r}\left[-\frac{E_1(r+x_9)}{2} - \frac{E_2(r-x_9)}{2}\right] & \equiv -\tfrac{E_1}{2}(1+\tfrac{x_9}{r}) - \tfrac{E_2}{2}(1-\tfrac{x_9}{r})\\
= -\frac{E_1+E_2}{2} - \frac{E_1-E_2}{2}\cos\theta & \equiv -E_{MICZ} + \tfrac{\Delta w^2}{4}\cos\theta.
\end{aligned}
$$

The analog of the 4th order anisotropic potential term for oscillator system [6] is in our 16D case the sum of 2 harmonic oscillators with identical in value but different in sign potential coefficients for nonlinearity of the 4th order ($b_1 = -b_2$). In the 9-dimensional space, this leads to the desired linear term:

$$
\begin{aligned}
b_1(u_su_s)^2 + b_2(v_sv_s)^2 &= b\left[(u_su_s)^2 - (v_sv_s)^2\right]\\
\to \frac{b}{r}\left[\frac{(r+x_9)^2}{4} - \frac{(r-x_9)^2}{4}\right] &= \tfrac{b}{4r}\left[4\ r\ x_9\right]\\
= b\ x_9 &\equiv br\cos\theta.
\end{aligned}
$$

We note in particular that in our case there is still a new term with a higher degree of nonlinearity than the one considered above.

5 Conclusion

Some generalization of 16D oscillator by anisotropic and nonlinear inharmonic terms and its dual analog for 9D related MICZ-Kepler systems by generalized version of the Kustaanheimo-Stiefel transformation was shown and analyzed.

The exact analytical solutions of the Schrödinger equation for abovementioned problems for QES class were discused and given for a few coordinates systems. It is shown that there is a correspondence in particular cases with similar results in lower dimensions.

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