Hypercomplex structures on tangent bundles

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With help of any metric connection $\nabla$ on an almost Hermitian manifold $M$ we can construct by the defined way an almost Hermitian hypercomplex structure on the tangent bundle $TM$. This structure includes two basic anticommutative almost-Hermitian structures for which introduced by the second author the fundamental tensor fields $h_1$ and $h_2$ are computed. It allows to consider various classes of almost-Hermitian hypercomplex structures on $TM$.

1. Introduction.

Let $(M, J, g)$ be an almost Hermitian manifold i.e. $J^2 = -I$ and $g(JX, JY) = g(X, Y)$ for $X, Y \in \chi(M)$, where $g$ is a fixed Riemannian metric on $M$.

For the Riemannian connection $\nabla$ the canonical connection $\nabla$ of the pair $(J, g)$ is defined by the formula

$$(1) \quad \nabla_X Y = \frac{1}{2}(\nabla X Y - J\nabla_X JY) = \nabla X Y + \frac{1}{2} J\nabla_X JY, \quad X, Y \in \chi(M).$$

The tensor field $h = \nabla - \nabla$ is called the second fundamental tensor field of the pair $(J, g)$ [1], in particular, we have $\nabla g = 0$, $\nabla J = 0$ and

$$(2) \quad h_X Y = -\frac{1}{2} \nabla_X (J) JY = \frac{1}{2}(\nabla_X Y + J\nabla_X JY),$$

$$(3) \quad h_{XYZ} = g(h_X Y, Z) = -h_{XZY}, \quad X, Y \in \chi(M).$$

The classification given in [3] has been rewritten in terms of the tensor field $h$ in [1].

Further, an almost Hermitian hypercomplex structure (ahhs) consists of $(J_1, J_2, J_3)$ where $J_i^2 = -I$, $J_1 J_2 = -J_2 J_1$, $g(J_i X, J_i Y) = g(X, Y)$, $i = 1, 2, 3$. For any Riemannian metric $\tilde{g}$ such a metric $g$ can be defined by the formula

$$g(X, Y) = \frac{1}{4}(\tilde{g}(X, Y) + \tilde{g}(J_1 X, J_1 Y) + \tilde{g}(J_2 X, J_2 Y) + \tilde{g}(J_3 X, J_3 Y)).$$

If $\nabla$ is the Riemannian connection of the metric $g$ then the canonical connection $\nabla$ of the ahhs has the following form

$$(4) \quad \nabla X Y = \frac{1}{4}(\nabla X Y - J_1 \nabla_X J_1 Y - J_2 \nabla_X J_2 Y - J_3 \nabla_X J_3 Y)$$

and $\nabla g = 0$, $\nabla J_i = 0$ for $i = 1, 2, 3$.

Proposition. Let $(M, J_1, g)$ be a Kaehlerian structure i.e. $\nabla J_1 = 0$ on $M$ then the connection given by (4) coincides with those defined by (1) for $(M, J_2, g)$ and $(M, J_3, g)$. In particular, the second fundamental tensor fields of $(M, J_2, g)$ and $(M, J_3, g)$ are the same.

Proof follow from (4) and (1) with help of condition $\nabla J_1 = 0$.

Theorem. A vector field $X$ is an infinitesimal isometry and an affine transformation with respect to $\nabla$ defined by (4) if and only if $L_X g = 0$ and $L_X h = 0$, where $h = \nabla - \nabla$, $L$ is the Lie differentiation with respect to $X$.

2. Hypercomplex structures on tangent bundles.
Let \((M, J, g)\) be an almost Hermitian manifold and \(TM\) be its tangent bundle. For a metric connection \(\tilde{\nabla} \ (\tilde{\nabla} g = 0)\) we consider the connection map \(\tilde{\nabla}\) of \(\nabla\) [2], defined by the formula

\[
\tilde{\nabla}_X Z = \tilde{K} \cdot X, \]

where \(Z\) is considered as a map from \(M\) into \(TM\) and we means by the right side the vector field on \(M\) assigning to \(p \in M\) the vector \(\tilde{K} \cdot X_p \in M_p\). If \(U \in TM\), we denote by \(H_U\) the kernel of \(\tilde{\nabla}_{|TM_U}\) and this 2n-dimensional subspace of \(TM_U\) is called the horizontal subspace of \(TM_U\). Let \(\pi\) denote the natural projection of \(TM\) onto \(M\) then \(\pi\) is a \(C^\infty\)-map of \(TTM\) onto \(TM\). If \(U \in TM\), we denote by \(V_U\) the kernel of \(\pi_{\mid TM_U}\) and this 2n-dimensional subspace of \(TM_U\) is called the vertical subspace of \(TM_U\) (\(\dim TM_U = 2 \dim M = 4n\)). The following maps are isomorphisms of corresponding vector spaces \((p = \pi(U))\).

\[
\pi_{\mid TM_U} : H_U \to M_p, \quad \tilde{\nabla}_{\mid TM_U} : V_U \to M_p
\]

and we have \(TM_U = H_U \oplus V_U\).

If \(X \in \chi(M)\) then there exists exactly one vector field on \(TM\) called the "horizontal lift" (resp. "vertical lift") of \(X\) and denoted by \(X^h\) (resp. \(X^v\)) such that for all \(U \in TM\):

\[
\pi_{\mid TM_U} X^h_U = X_{\pi(U)}^h, \quad \pi_{\mid TM_U} X^v_U = O_{\pi(U)}, \quad \tilde{\nabla} X^h_U = O_{\pi(U)}, \quad \tilde{\nabla} X^v_U = X_{\pi(U)}.
\]

Let \(\tilde{\nabla}\) be the curvature tensor field of \(\nabla\) then following [2] we have

\[
[X^v, Y^v] = 0, \quad [X^h, Y^v] = \nabla_X Y, \quad \pi_{\mid TM_U} ([X^h, Y^h]_U) = [X, Y],
\]

\[
\tilde{\nabla} ([X^h, Y^h]_U) = \tilde{\nabla}(X, Y) U.
\]

For vector fields \(\overline{X} = X^h \oplus X^v\) and \(\overline{Y} = Y^h \oplus Y^v\) on \(TM\) the natural Riemannian metric \(<,>\) is defined on \(TM\) by the formula

\[
<\overline{X}, \overline{Y}> = g(\pi_{\mid TM_U} \overline{X}, \pi_{\mid TM_U} \overline{Y}) + g(\tilde{\nabla} \overline{X}, \tilde{\nabla} \overline{Y}).
\]

It is clear that the subspaces \(H_U\) and \(V_U\) are orthogonal with respect to \(<,>\).

1). We define a tensor field \(J_1\) on \(TM\) by the equalities

\[
J_1 X^h = X^v, \quad J_1 X^v = -X^h, \quad X \in \chi(M).
\]

It is easy to verify that \((TM, J_1, <,>)\) is an almost Hermitian manifold.

Remark. This construction uses only the Riemannian metric \(g\) and does not depend on the almost complex structure \(J\).

Let \(h^1\) be the second fundamental tensor field of the pair \((J_1, <,>)\), see (2), (3). We have obtained the following cases for the tensor field \(h^1\) assuming all the vector fields to be orthonormal

1.1°) \(h^1_{X^h Y^h Z^h} = \frac{1}{4} \left( g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_X Y, \tilde{\nabla}_X Y) \right); \)

2.1°) \(h^1_{X^h Y^h Z^v} = -\frac{1}{4} \left( g(\tilde{\nabla}(X, Y) Z, U) + g(\tilde{\nabla}(Z, X) Y, U) \right); \)
3.1°) \[ h_{X,Y,Z}^1 = -\frac{1}{4} \left( g(\tilde{R}(Z,X)Y, U) + g(\tilde{R}(X,Y)Z, U) \right); \]
4.1°) \[ h_{X,Y,Z}^2 = -\frac{1}{4} g(\tilde{R}(Z,Y)X, U); \]
5.1°) \[ h_{X,Y,Z}^3 = \frac{1}{4} g(\tilde{R}(Z,Y)X, U); \]
6.1°) \[ h_{X,Y,Z}^4 = 0; \]
7.1°) \[ h_{X,Y,Z}^5 = 0; \]
8.1°) \[ h_{X,Y,Z}^6 = \frac{1}{2} \left( g(\nabla X Y, Z) - g(\nabla Y Z, X) \right). \]

Thus, the tensor field \( h^i \) (class of the structure \( (J_1, <, >) \)) strongly depends on the connection \( \nabla \).

II). We define a tensor field \( J_2 \) on \( TM \) assuming
\[ J_2 X^h = (JX)^h, \quad J_2 X^v = -(JX)^v, \quad X \in \chi(M). \]

One can verify that \( (TM, J_1, J_2, J_3 = J_1 J_2, <, >) \) is an almost Hermitian hypercomplex manifold.

Let \( h^j \) be the second fundamental tensor field of the pair \( (J_2, <, >) \), see (2), (3).
Assuming all the vector fields to be orthonormal we have got
1.2°) \[ h_{X,Y,Z}^1 = h_{X,Y,Z}; \]
2.2°) \[ h_{X,Y,Z}^2 = -\frac{1}{4} \left( g(\tilde{R}(X,Y)Z, U) + g(\tilde{R}(X,Z)Y, U) \right); \]
3.2°) \[ h_{X,Y,Z}^3 = \frac{1}{4} \left( g(\tilde{R}(X,Y)Z, U) + g(\tilde{R}(X,Z)Y, U) \right); \]
4.2°) \[ h_{X,Y,Z}^4 = -\frac{1}{4} \left( g(\tilde{R}(X,Y)Z, U) + g(\tilde{R}(X,Z)Y, U) \right); \]
5.2°) \[ h_{X,Y,Z}^5 = 0; \]
6.2°) \[ h_{X,Y,Z}^6 = 0; \]
7.2°) \[ h_{X,Y,Z}^7 = 0; \]
8.2°) \[ h_{X,Y,Z}^8 = \frac{1}{2} \left( g(\nabla X Y, Z) - g(\nabla Y Z, X) \right). \]

It is clear that the construction of the aHhs on \( TM \) strongly depends on the connection \( \nabla \) and we can obtain in this way an infinite dimensional set of aHhs.

Using the remark in I), and the arguments above we have got the following

**Theorem.** Let \((M, J, h)\) be a Riemannian manifold. Then there exists an infinite dimensional set of aHhs on \( TTM \). This structures can be constructed by the method above.

**References**


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