Hypercomplex structures on tangent bundles

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With help of any metric connection $\widetilde{\nabla}$ on an almost Hermitian manifold M w can construct by the defined way an almost Hermitian hypercomplex structure o the tangent bundle TM. This structure includes two basic anticommutative almos Hermitian structures for which introduced by the second author the fundament tensor fields h^1 and h^2 are computed. It allows to consider various classes of almo Hermitian hypercomplex structures on TM.

1. Introduction.

Let (M, J, g) be an almost Hermitian manifold i.e. $J^2 = -I$ and $g(JX, JY) = g(X,Y)$ for $X, Y \in \chi(M)$, where g is a fixed Riemannian metric on M.

For the Riemannian connection ∇ the canonical connection $\overline{\nabla}$ of the pair $(J, g$ [1] is defined by the formula

(1) $\quad \overline{\nabla}_X Y = \frac{1}{2}(\nabla_X Y - J\nabla_X JY) = \nabla_X Y + \frac{1}{2}\nabla_X(J)JY, \qquad X, Y \in \chi(M).$

The tensor field $h = \nabla - \overline{\nabla}$ is called the second fundamental tensor field of the pair (J, g) [1], in particular, we have $\overline{\nabla} g = 0$, $\overline{\nabla} J = 0$ and

(2) $\quad h_X Y = -\frac{1}{2}\nabla_X(J)JY = \frac{1}{2}(\nabla_X Y + J\nabla_X JY),$

(3) $\quad h_{XYZ} = g(h_X Y, Z) = -h_{XZY}, \qquad X, Y \in \chi(M).$

The classification given in [3] has been rewritten in terms of the tensor field h in [1].

Further, an almost Hermitian hypercomplex structure (aHhs) consists of $(J_1, J_2, J_3$ where $J_i^2 = -I$, $J_1J_2 = -J_2J_1 = J_3$, $g(J_iX, J_iY) = g(X,Y)$, $i = 1,2,3$. For any Riemannian metric $\widetilde{g}$ such a metric g can be defined by the formula

$$g(X,Y) = \frac{1}{4}\left(\widetilde{g}(X,Y) + \widetilde{g}(J_1X, J_1Y) + \widetilde{g}(J_2X, J_2Y) + \widetilde{g}(J_3X, J_3Y)\right).$$

If ∇ is the Riemannian connection of the metric g then the canonical connection $\overline{\nabla}$ of the aHhs has the following form

(4) $\quad \overline{\nabla}_X Y = \frac{1}{4}\left(\nabla_X Y - J_1\nabla_X J_1 Y - J_2\nabla_X J_2 Y - J_3\nabla_X J_3 Y\right)$

and $\overline{\nabla} g = 0$, $\overline{\nabla} J_i = 0$ for $i = 1, 2, 3$.

Proposition. *Let (M, J_1, g) be a Kaehlerian structure i.e. $\nabla J_1 = 0$ on M then the connection given by (4) coincides with those defined by (1) for (M, J_2, g) and (M, J_3, g). In particular, the second fundamental tensor fields of (M, J_2, g) and (M, J_3, g) are the same.*

Proof follow from (4) and (1) with help of condition $\nabla J_1 = 0$.

Theorem. *A vector field X is an infinitesimal isometry and an affine transformation with respect to $\overline{\nabla}$ defined by (4) if and only if $L_X g = 0$ and $L_X h = 0$, where $h = \nabla - \overline{\nabla}$, L is the Lie differentiation with respect to X.*

2. Hypercomplex structures on tangent bundles.

Let (M, J, g) be an almost Hermitian manifold and TM be its tangent bundle. For a metric connection $\widetilde{\nabla}$ $\left(\widetilde{\nabla} g = 0\right)$ we consider the connection map $\widetilde{K}$ of $\widetilde{\nabla}$ [2], defined by the formula

$$\widetilde{\nabla}_X Z = \widetilde{K} Z_* X,$$

where Z is considered as a map from M into TM and we means by the right side the vector field on M assigning to $p \in M$ the vector $\widetilde{K} Z_* X_p \in M_p$. If $U \in TM$, we denote by H_U the kernel of $\widetilde{K}_{|TM_U}$ and this $2n$-dimensional subspace of TM_U is called the horizontal subspace of TM_U. Let π denote the natural projection of TM onto M then π_* is a C^∞-map of TTM onto TM. If $U \in TM$, we denote by V_U the kernel of $\pi_{*|TM_U}$ and this $2n$-dimensional subspace of TM_U is called the vertical subspace of TM_U ($\dim TM_U = 2 \dim M = 4n$). The following maps are isomorphisms of corresponding vector spaces ($p = \pi(U)$).

$$\pi_{*|TM_U} : H_U \to M_p, \qquad \widetilde{K}_{|TM_U} : V_U \to M_p$$

and we have $TM_U = H_U \oplus V_U$.

If $X \in \chi(M)$ then there exists exactly one vector field on TM called the "horizontal lift" (resp. "vertical lift") of X and denoted by X^h (resp. X^v) such that for all $U \in TM$:

$$\pi_* X^h_U = X_{\pi(U)}, \qquad \pi_* X^v_U = O_{\pi(U)}; \qquad \widetilde{K} X^h_U = O_{\pi(U)}, \qquad \widetilde{K} X^v_U = X_{\pi(U)}.$$

Let $\widetilde{R}$ be the curvature tensor field of $\widetilde{\nabla}$ then following [2] we have

$$[X^v, Y^v] = 0, \qquad [X^h, Y^v] = (\widetilde{\nabla}_X Y)^v, \qquad \pi_*([X^h, Y^h]_U) = [X, Y],$$
$$\widetilde{K}([X^h, Y^h]_U) = \widetilde{R}(X, Y)U.$$

For vector fields $\overline{X} = X^h \oplus X^v$ and $\overline{Y} = Y^h \oplus Y^v$ on TM the natural Riemannian metric $<,>$ is defined on TM by the formula

$$< \overline{X}, \overline{Y} >= g(\pi_* \overline{X}, \pi_* \overline{Y}) + g(\widetilde{K}\overline{X}, \widetilde{K}\overline{Y}).$$

It is clear that the subspaces H_U and V_U are orthogonal with respect to $<,>$.

I). We define a tensor field J_1 on TM by the equalities

$$J_1 X^h = X^v, \qquad J_1 X^v = -X^h, \qquad X \in \chi(M).$$

It is easy to verify that $(TM, J_1, <,>)$ is an almost Hermitian manifold.

Remark. *This construction uses only the Riemannian metric g and does not depend on the almost complex structure J.*

Let h^1 be the second fundamental tensor field of the pair $(J_1, <,>)$, see (2), (3). We have obtained the following cases for the tensor field h^1 assuming all the vector fields to be orthonormal

1.1^0) $\quad h^1_{X^h Y^h Z^h} = \frac{1}{2}\left(g(\nabla_X Y, Z) - g(\widetilde{\nabla}_X Y, Z)\right);$

2.1^0) $\quad h^1_{X^h Y^h Z^v} = -\frac{1}{4}\left(g(\widetilde{R}(X, Y)Z, U) + g(\widetilde{R}(Z, X)Y, U)\right);$

3.1⁰) $h^1_{X^hY^vZ^h} = -\frac{1}{4}\Big(g(\widetilde{R}(Z,X)Y,U) + g(\widetilde{R}(X,Y)Z,U)\Big)$;

4.1⁰) $h^1_{X^vY^hZ^h} = -\frac{1}{4}g(\widetilde{R}(Z,Y)X,U)$;

5.1⁰) $h^1_{X^vY^vZ^v} = \frac{1}{4}g(\widetilde{R}(Z,Y)X,U)$;

6.1⁰) $h^1_{X^vY^vZ^h} = 0$;

7.1⁰) $h^1_{X^vY^hZ^v} = 0$;

8.1⁰) $h^1_{X^hY^vZ^v} = \frac{1}{2}\Big(g(\widetilde{\nabla}_X Y,Z) - g(\nabla_X Y,Z)\Big)$.

Thus, the tensor field h^1 (class of the structure $(J_1,<,>)$) strongly depends on the connection $\widetilde{\nabla}$.

II). We define a tensor field J_2 on TM assuming

$$J_2X^h = (JX)^h, \qquad J_2X^v = -(JX)^v, \qquad X \in \chi(M).$$

One can verify that $(TM, J_1, J_2, J_3 = J_1J_2, <,>)$ is an almost Hermitian hypercomplex manifold.

Let h^2 be the second fundamental tensor field of the pair $(J_2,<,>)$, see (2), (3). Assuming all the vector fields to be orthonormal we have got

1.2⁰) $h^2_{X^hY^hZ^h} = h_{XYZ}$;

2.2⁰) $h^2_{X^hY^hZ^v} = -\frac{1}{4}\Big(g(\widetilde{R}(X,Y)Z,U) + g(\widetilde{R}(X,JY)JZ,U)\Big)$;

3.2⁰) $h^2_{X^hY^vZ^h} = \frac{1}{4}\Big(g(\widetilde{R}(X,Z)Y,U) + g(\widetilde{R}(X,JZ)JY,U)\Big)$;

4.2⁰) $h^2_{X^vY^hZ^h} = -\frac{1}{4}\Big(g(\widetilde{R}(Z,Y)X,U) + g(\widetilde{R}(JZ,JY)X,U)\Big)$;

5.2⁰) $h^2_{X^vY^vZ^v} = 0$;

6.2⁰) $h^2_{X^vY^vZ^h} = 0$;

7.2⁰) $h^2_{X^vY^hZ^v} = 0$;

8.2⁰) $h^2_{X^hY^vZ^v} = \frac{1}{2}\Big(g(\widetilde{\nabla}_X Y,Z) - g(\widetilde{\nabla}_X JY,JZ)\Big)$.

It is clear that the construction of the aHhs on TM strongly depends on the connection $\widetilde{\nabla}$ and we can obtain in this vay an infinite dimensional set of aHhs.

Using the *remark* in I) and the arguments above we have got the following

Theorem. *Let (M,g) be a Riemannian manifold. Then there exists an infinite dimensional set of aHhs on TTM. This structures can be constructed by the method above.*

References

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