Nonlinear Deformation of Dressing Chain

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Abstract

Algebra of the operators connected with the nonlinear dressing chain is considered. Some realizations of given algebra via the generators of the quantum algebra, which corresponds to a nonlinear deformation of $SU(2)$ and $SU_q(2)$ are proposed.
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The linear dressing chain and spectral theory of the Schrödinger operator were considered in [1]-[2]. It was shown that the finite-gap theory and the generalization of the harmonic oscillator with higher order creation-annihilation operators are closely connected. The goal of this work is to consider the development of this method for the nonlinear dressing chain.

First we recall briefly the formulation of the problem. It is well known that the one dimensional Schrödinger equation

$$L\psi(x) = (-D^2 + u(x))\,\psi(x) = \lambda\psi(x)$$

may be written such as the sequences of the hamiltonians L_j by the creation-annihilation operators $a_j^{\pm}$ in the next type:

$$L_j = a_j^+ a_j^- = (-D + f_j)(D + f_j)\,, \quad j = 0\,, \pm 1\,, \pm 2 \ldots \tag{1}$$

The linear dressing chain is determined by chain of differential-difference equations for functions f_j

$$D(f_j + f_{j+1}) = f_j^2 - f_{j+1}^2 + \lambda_j$$

or, at the operator level, by the abstract factorization chain

$$a_{j+1}^+ a_{j+1}^- = a_j^- a_j^+ + \lambda_j\,, \tag{2}$$

where $D = d/dx$ and λ_j are some constants.

One can also note that the following interwining relations are satisfying automatically from (1)-(2):

$$L_j a_j^+ = a_j^+ L_{j+1}\,; \qquad a_j^- L_j = L_{j+1} a_j^-\,. \tag{3}$$

Among the various reductions of the chain (2) there is a general periodic closure:

$$L_{j+N} = qTL_jT^{-1} + \mu\,, \tag{4}$$

where T is some invertable operator [2]. So, we get the next symmetry polynomial algebra:

$$\begin{aligned} Hb^+ &= b^+(qH+\mu)\,;\\ b^-H &= (qH+\mu)b^-\,;\\ b^+b^- &= \prod_{k=0}^{k=N-1}(H-\lambda_{j+k})\,;\\ b^-b^+ &= \prod_{k=0}^{k=N-1}(qH+\mu-\lambda_{j+k}) \end{aligned} \qquad (5)$$

in realizations of operators

$$\begin{aligned} H &= L_j\,;\\ b^+ &= a_j^+\ldots a_{j+N-1}^+T^+\,;\\ b^- &= T^{-1}a_{j+N-1}^-\ldots a_j^-\,. \end{aligned} \qquad (6)$$

In the present paper I would like to stress two moments. The former deals with the determination of the hamiltonians L_j and L_{j+1} (1), (2) or the interwined relations (3). The later is connected the periodic closure (4). Therefore, I propose the next generalization of them:

$$\begin{aligned} L_jA_j^+ &= A_j^+g_j(L_{j+1})\,;\\ A_j^-L_j &= g_j(L_{j+1})A_j^-\,;\\ L_{j+N} &= TP_j(L_j)T^{-1}\,, \end{aligned} \qquad (7)$$

where $g_j(x)$, $P_j(x)$ are arbitrary functions of its arguments x and T is arbitrary invertable operator. Let us introduce the operators:

$$\begin{aligned} K &= L_j\,;\\ B^+ &= A_j^+\ldots A_{j+N-1}^+T\,;\\ B^- &= T^{-1}A_{j+N-1}^-\ldots A_j^-\,. \end{aligned} \qquad (8)$$

Using (7), (8) we may to build the following nonlinear algebra:

$$\begin{aligned} KB^+ &= g_{j,j+N-1}(P_j(K))B^+\,;\\ B^-K &= B^-g_{j,j+N-1}(P_j(K))\,;\\ B^-B^+ &= \prod_{k=0}^{k=N-1}g_{j+k,j+N-1}(P_j(K))\,;\\ B^+B^- &= \prod_{k=0}^{k=N-1}g_{j,j+k-1}^{-1}(K)\,, \end{aligned} \qquad (9)$$

where

$$g^{-1}(x)\neq\frac{1}{g(x)}\,;\quad g^{-1}(g(x))=x\,;$$

$$g_{j,k}(x)=\begin{cases} x, & j>k\\ g_j(x), & j=k\\ g_j(g_{j+1}(\ldots g_k(x)\ldots)), & j<k \end{cases}$$

Due to the freedom in choice of the functions $g_j(x)$, $P_j(x)$ the algebra can take various forms. According to [3] algebra (9) may be considered as the detailization of form for nonlinear deformation $SU(2)$. Then one found that there are only two cases with the analytical formula for any values j,k in complex functions $g_{j,k}(x)$. It is surprising that these cases deal with the arithmetical and the geometrical progressions. In the first one we have $g_j(x)=c_jx+d_j$ and call this the linear dressing chain. The more detailed examination of this will be given in the next paper. Now we represent some results for nonlinear dressing chain. The second case is defined as follows:

$$g_j(x) = u_j x^{v_j}\,, \qquad P_j(x) = p_j x + r_j\,.$$

Substituting these explicit forms of the functions $g_j(x)\,, P_j(x)$ into (9) we obtain the next nonlinear symmetry algebra:

$$\begin{aligned} KB^+ &= B^+Q(pK+r)^R\,;\\ B^-K &= Q(pK+r)^R B^-\,;\\ B^+B^- &= \prod_{k=0}^{k=N-1} Z_k K^{W_k}\,;\\ B^-B^+ &= \prod_{k=0}^{k=N-1} X_k (pK+r)^{Y_k}, \end{aligned} \tag{10}$$

where the index $j = 0$ was fixed for simplicity of notation. Also in (10) we use the following notations for structure constants of algebra:

$$V_{\pm,k} = \begin{cases} 1, & k < 0,\\ \prod_{l=0}^{l=k} v_l^{\pm 1}, & k \geq 0. \end{cases}$$

$$Q = U_{+,N-1}\,, \qquad R = V_{+,N-1} \qquad X_k = \frac{\varrho}{U_{+k-1}}$$

$$Y_k = \frac{R}{V_{+,k-1}} \qquad Z_k = U_{-,k-1}\,, \qquad W_k = V_{-k-1}$$

$$U_{\pm,k} = \begin{cases} 1, & k < 0\\ \prod_{l=0}^{l=k} u_l^{\pm V_{\pm,l-1}}, & k \geq 0 \end{cases}$$

Note that if we choose $v_j = 1$, then this is the particular case of the linear dressing chain. Let us consider more interesting example. For new operator $M = \ln K$ and $r = 0$ we obtain next algebra of the operators M, $B^\pm$, which may be considered as the generalization for $SU_q(2)$:

$$MB^+ = B^+(RM + C)\,; \qquad B^-M = (RM + C)B^-\,;$$
$$B^-B^+ = \prod_{k=0}^{k=N-1} Z_k e^{W_k M}\,, \quad B^+B^- = \prod_{k=0}^{k=N-1} X_k p^{f_k} e^{Y_k M}\,,$$

where $C = \ln Q + R \ln p$.
To conclude, we proposed nonlinear generalization of the dressing chain and found symmetry algebra of the operators connected with them. Realizaion of given algebra via the generators of the quantum algebra, which corresponds to a nonlinear deformation of $SU(2)$ and $SU_q(2)$ was given.
There are some problems of general nonlinear chain those were not discussed here. One of them is related to the consideration of the one-dimensional relaivistic Schrödinger equations [4]. Another problems are the physical realizations or supersymmetrical structures of the constructions (7)-(9) and [5].

References

[1] Veselov *A P* and Shabat *A B* 1993 *Funkt. Anal. i ego Pril.* **27** 1–27

[2] Spiridonov *V* 1995 *Phys. Rev.* **52** 1909–1935

[3] Delbecq *C* and Quesne *C* 1993 *J. Phys. A: Math. Gen.* **26** L127–L134

[4] Mir–Kasimov *R M* 1991 *J. Phys. A: Math. Gen.* **24** 4283–4302

[5] Rocek *M* 1991 *Phys. Lett.* **B255** 554–557