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TWO-DIMENSIONAL STATIONARY TEMPERATURE FIELD OF A SYSTEM OF BOUNDED INHOMOGENEOUS CYLINDERS WHICH ARE IN IDEAL THERMAL CONTACT

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The problem on uniform stationary lateral heating of two bounded cylindrical bodies with different thermophysical characteristics which are in ideal thermal contact within the region of contact has been considered. The exact analytical solution of this problem has been obtained and the conditions of energy balance have been investigated.

In recent years, in connection with the progress made in high-temperature thermal physics, thermal problems with boundary conjugation conditions have acquired great importance. However, an analytical solution of these problems, especially in nonstationary and bounded cases, involves certain mathematical difficulties. Despite the great number of existing solutions of such kind [1–4], they are usually attributed to specific assumptions (one-dimensionality, unboundedness or semiboundedness of space, selection of a special plane or axis). In the present work, we consider the problem on spatial distribution of the temperature field of two inhomogeneous bounded cylinders when they are under stationary conditions of heating.

There are two cylinders of radius R and lengths l_1 and l_2 with different thermophysical characteristics. The cylinders are in ideal thermal contact in the plane $z = 0$ and from the side surface they are heated by a constant heat flux with surface power Q_R. It is necessary to find the distribution of the stationary thermal field in the cylinders. On the ends we have heat exchange with the environment following the Newton law with heat-transfer coefficients α_1 and α_2. If we assume that the environment temperature is equal to zero, then for the stationary temperatures of the cylinders we obtain the equations

$$\frac{\partial^2 T_1}{\partial z^2} + \frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} = 0\,, \quad -l_1 < z < 0\,; \quad \frac{\partial^2 T_2}{\partial z^2} + \frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r} = 0\,, \quad 0 < z < l_2\,, \tag{1}$$

with the boundary conditions

$$\lambda_1 \frac{\partial T_1(r, -l_1)}{\partial z} = \alpha_1 T_1(r, -l_1)\,, \quad \lambda_2 \frac{\partial T_2(r, l_2)}{\partial z} = -\alpha_2 T_2(r, l_2)\,, \tag{2}$$

$$\lambda_1 \frac{\partial T_1(R, z)}{\partial r} = \lambda_2 \frac{\partial T_2(R, z)}{\partial r} = Q_R \tag{3}$$

and the conjugation conditions

$$\lambda_1 \frac{\partial T_1(r, 0)}{\partial z} = \lambda_2 \frac{\partial T_2(R, 0)}{\partial z}\,, \quad T_1(r, 0) = T_2(r, 0)\,. \tag{4}$$

We apply the finite Hankel integral transform to problem (1)–(4):

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$$\overline{T}_i(\mu_m, z) = \int_0^R rJ_0\left(\frac{\mu_m}{R} r\right) T(r, z)\, dr\,, \tag{5}$$

where μ_m are the roots of the equation $J_1(\mu) = 0$. Here, the differential operator $\frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r}\frac{\partial T_i}{\partial r}$ takes the form

$$RJ_0(\mu_m)\frac{\partial T_i(R, z)}{\partial r} - \left(\frac{\mu_m}{R}\right)^2 \overline{T}_i(\mu_m, z) = RJ_0(\mu_m)\frac{Q_R}{\lambda_i} - \gamma_m^2 \overline{T}_i(\mu_m, z)\,, \tag{6}$$

in which $\gamma_m = \mu_m/R$. Finally, from problem (1)–(4) we come to the following problem:

$$\frac{d^2\overline{T}_1}{dz^2} - \gamma_m^2\overline{T}_1 = -\frac{RJ_0(\mu_m)\, Q_R}{\lambda_1}\,, \quad -l_1 < z < 0\,; \quad \frac{d^2\overline{T}_2}{dz^2} - \gamma_m^2\overline{T}_2 = -\frac{RJ_0(\mu_m)\, Q_R}{\lambda_2}\,, \quad 0 < z < l_2\,, \tag{7}$$

with the boundary conditions

$$\lambda_1\frac{d\overline{T}_1(\mu_m, -l_1)}{dz} = \alpha_1\overline{T}_1(\mu_m, -l_1)\,, \quad \lambda_2\frac{d\overline{T}_2(\mu_m, l_2)}{dz} = -\alpha_2\overline{T}_2(\mu_m, l_2) \tag{8}$$

and the conjugation conditions

$$\lambda_1\frac{d\overline{T}_1(\mu_m, 0)}{dz} = \lambda_2\frac{d\overline{T}_2(\mu_m, 0)}{dz}\,, \quad \overline{T}_1(\mu_m, 0) = \overline{T}_2(\mu_m, 0)\,. \tag{9}$$

The general solutions of Eqs. (7) and (8) are

$$\overline{T}_1(\mu_m, z) = A\cosh\gamma_m z + B\sinh\gamma_m z + \frac{RJ_0(\mu_m)\, Q_R}{\gamma_m^2\lambda_1}\,, \tag{10}$$

$$\overline{T}_2(\mu_m, z) = C\cosh\gamma_m z + D\sinh\gamma_m z + \frac{RJ_0(\mu_m)\, Q_R}{\gamma_m^2\lambda_2}\,. \tag{11}$$

To determine the constants A, B, C, and D we use boundary conditions (8) and conjugation conditions (9), which yields

$$A = \frac{RJ_0(\mu_m)\, Q_R}{\gamma_m^2}\left\{\left(1 - \frac{\lambda_2}{\lambda_1}\right)(\gamma_m\cosh\gamma_m l_1 + h_1\sinh\gamma_m l_1)(\gamma_m\sinh\gamma_m l_2 + h_2\cosh\gamma_m l_2) - \right.$$

$$\left. - h_2(\gamma_m\cosh\gamma_m l_1 + h_1\sinh\gamma_m l_1) - h_1(\gamma_m\cosh\gamma_m l_2 + h_2\sinh\gamma_m l_2)\right\}\left\{\lambda_1(\gamma_m\sinh\gamma_m l_1 + h_1\cosh\gamma_m l_1)\times\right.$$

$$\left.\times(\gamma_m\cosh\gamma_m l_2 + h_2\sinh\gamma_m l_2) + \lambda_2(\gamma_m\cosh\gamma_m l_1 + h_1\sinh\gamma_m l_1)(\gamma_m\sinh\gamma_m l_2 + h_2\cosh\gamma_m l_2)\right\}^{-1}, \tag{12}$$

$$B = \frac{RJ_0(\mu_m)\, Q_R}{\gamma_m^2}\left\{\left(1 - \frac{\lambda_2}{\lambda_1}\right)(\gamma_m\sinh\gamma_m l_1 + h_1\cosh\gamma_m l_1)(\gamma_m\sinh\gamma_m l_2 + h_2\cosh\gamma_m l_2) - \right.$$

$$-h_2(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1) + h_1 \frac{\lambda_2}{\lambda_1}(\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2)\Bigg]\Big\{\lambda_1(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1)\times$$

$$\times(\gamma_m \cosh \gamma_m l_2 + h_2 \sinh \gamma_m l_2) + \lambda_2(\gamma_m \cosh \gamma_m l_1 + h_1 \sinh \gamma_m l_1)(\gamma_m \cosh \gamma_m l_1 + h_1 \sinh \gamma_m l_1)\Big\}^{-1}. \quad (13)$$

$$C = \frac{RJ_0(\mu_m)\,Q_R}{\gamma_m^2}\Bigg[\left(1 - \frac{\lambda_1}{\lambda_2}\right)(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1)(\gamma_m \cosh \gamma_m l_2 + h_2 \sinh \gamma_m l_2) -$$

$$-h_2(\gamma_m \cosh \gamma_m l_1 + h_1 \sinh \gamma_m l_1) - h_1(\gamma_m \cosh \gamma_m l_2 + h_2 \sinh \gamma_m l_2)\Bigg]\Big\{\lambda_1(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1)\times$$

$$\times(\gamma_m \cosh \gamma_m l_2 + h_2 \sinh \gamma_m l_2) + \lambda_2(\gamma_m \cosh \gamma_m l_1 + h_1 \sinh \gamma_m l_1)(\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2)\Big\}^{-1}. \quad (14)$$

$$D = \frac{RJ_0(\mu_m)\,Q_R}{\gamma_m^2}\Bigg[\left(\frac{\lambda_1}{\lambda_2} - 1\right)(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1)(\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2) -$$

$$-h_2\frac{\lambda_1}{\lambda_2}(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1) + h_1(\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2)\Bigg]\Big\{\lambda_1(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1)\times$$

$$\times(\gamma_m \cosh \gamma_m l_2 + h_2 \sinh \gamma_m l_2) + \lambda_2(\gamma_m \cosh \gamma_m l_1 + h_1 \sinh \gamma_m l_1)(\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2)\Big\}^{-1}. \quad (15)$$

where $h_i = \alpha_i/\lambda_i$.

We emphasize here that the results obtained hold for all $\mu_m > 0$ which are the roots of the equation $J_1(\mu) = 0$. But the function $J_1(\mu)$ has zero at the point $\mu = 0$ as well. The solution of problem (1)–(4) for this case will be considered separately. The Hankel transform and the operator of differentiation with respect to r will take the forms $\bar{T}_i(0, z) = \int_0^R rT_i(r, z)\,dr$ and $\frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r}\frac{\partial T_i}{\partial r} = R\frac{Q_R}{\lambda_i}$ respectively. Thus, in the case of the zero root, we go from problem (1)–(4) to the following problem:

$$\frac{d^2\bar{T}_1}{dz^2} = -\frac{RQ_R}{\lambda_1}, \quad -l_1 < z < 0; \quad \frac{d^2\bar{T}_2}{dz^2} = -\frac{RQ_R}{\lambda_2}, \quad 0 < z < l_2, \quad (16)$$

with the boundary conditions

$$\lambda_1 \frac{d\bar{T}_1(0, -l_1)}{dz} = \alpha_1 \bar{T}_1(0, -l_1), \quad \lambda_2 \frac{d\bar{T}_2(0, l_2)}{dz} = -\alpha_2 \bar{T}_2(0, l_2) \quad (17)$$

and the conjugation conditions

$$\lambda_1 \frac{d\bar{T}_1(0, 0)}{dz} = \lambda_2 \frac{d\bar{T}_2(0, 0)}{dz}, \quad \bar{T}_1(0, 0) = \bar{T}_2(0, 0). \quad (18)$$

The general solutions of Eqs. (16) are as follows:

$$\bar{T}_1(0, z) = -\frac{RQ_R}{2\lambda_1} z^2 + Ez + F, \quad \bar{T}_2(0, z) = -\frac{RQ_R}{2\lambda_2} z^2 + Gz + H. \tag{19}$$

To determine the constants E, F, G, and H we use conditions (17) and (18), which yields

$$E = RQ_R \frac{h_1 l_2 \left(1 + \frac{h_2 l_2}{2}\right) - \frac{\lambda_2}{\lambda_1} h_2 l_1 \left(1 + \frac{h_1 l_1}{2}\right)}{\lambda_1 h_1 (1 + h_2 l_2) + \lambda_2 h_2 (1 + h_1 l_1)}, \tag{20}$$

$$G = RQ_R \frac{\frac{\lambda_1}{\lambda_2} h_1 l_2 \left(1 + \frac{h_2 l_2}{2}\right) - h_2 l_1 \left(1 + \frac{h_1 l_1}{2}\right)}{\lambda_1 h_1 (1 + h_2 l_2) + \lambda_2 h_2 (1 + h_1 l_1)}, \tag{21}$$

$$F = H = RQ_R \frac{l_1 (1 + h_2 l_2) \left(1 + \frac{h_1 l_1}{2}\right) + l_2 (1 + h_1 l_1) \left(1 + \frac{h_2 l_2}{2}\right)}{\lambda_1 h_1 (1 + h_2 l_2) + \lambda_2 h_2 (1 + h_1 l_1)}. \tag{22}$$

Now, with account for Eqs. (10)–(15) and (19)–(22) and by means of the inverse finite Hankel transform $T_i(r, z) = \frac{2}{R^2}\left(\bar{T}_i(0, z) + \sum_{m=1}^{\infty} \frac{J_0\left(\mu_m \frac{r}{R}\right)}{J_0^2(\mu_m)} \bar{T}_i(\mu_m, z)\right)$ we obtain the final solution of the initial problem (1)–(4):

$$T_1(r, z) =$$

$$= \frac{2Q_R}{R}\left(\frac{l_1 (1 + h_2 l_2)\left(1 + \frac{h_1 l_1}{2}\right) + l_2 (1 + h_1 l_1)\left(1 + \frac{h_2 l_2}{2}\right) + \left(h_1 l_2 \left(1 + \frac{h_2 l_2}{2}\right) - \frac{\lambda_2}{\lambda_1} h_2 l_1 \left(1 + \frac{h_1 l_1}{2}\right)\right) z}{\lambda_1 h_1 (1 + h_2 l_2) + \lambda_2 h_2 (1 + h_1 l_1)} - \right.$$

$$- \frac{z^2}{2\lambda_1} + \sum_{m=1}^{\infty} \frac{J_0\left(\mu_m \frac{r}{R}\right)}{\gamma_m^2 J_0(\mu_m)} \left[\left\langle \left(1 - \frac{\lambda_2}{\lambda_1}\right)(\gamma_m \cosh \gamma_m l_1 + h_1 \sinh \gamma_m l_1)(\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2) - \right.\right.$$

$$\left. - h_2 (\gamma_m \cosh \gamma_m l_1 + h_1 \sinh \gamma_m l_1) - h_1 (\gamma_m \cosh \gamma_m l_2 + h_2 \sinh \gamma_m l_2) \right\rangle \cosh \gamma_m z +$$

$$+ \left\langle \left(1 - \frac{\lambda_2}{\lambda_1}\right)(\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1)(\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2) - \right.$$

$$\left.\left. - h_2 (\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1) + h_1 \frac{\lambda_2}{\lambda_1} (\gamma_m \sinh \gamma_m l_2 + h_2 \cosh \gamma_m l_2) \right\rangle \sinh \gamma_m z \right] \times$$

$$\times \left[\lambda_1 (\gamma_m \sinh \gamma_m l_1 + h_1 \cosh \gamma_m l_1)(\gamma_m \cosh \gamma_m l_2 + h_2 \sinh \gamma_m l_2) + \right.$$

$$+\lambda_2(\gamma_m\cosh\gamma_m l_1 + h_1\sinh\gamma_m l_1)(\gamma_m\sinh\gamma_m l_2 + h_2\cosh\gamma_m l_2)\Big]^{-1} + \frac{1}{\lambda_1}\Bigg\}\Bigg), \quad -l_1 \le z \le 0; \tag{23}$$

$$T_2(r,z) = \frac{2Q_R}{R}\left(\frac{l_1(1+h_2l_2)\left(1+\frac{h_1l_1}{2}\right) + l_2(1+h_1l_1)\left(1+\frac{h_2l_2}{2}\right) + \left(\frac{\lambda_1}{\lambda_2}h_1l_2\left(1+\frac{h_2l_2}{2}\right) - h_2l_1\left(1+\frac{h_1l_1}{2}\right)\right)z}{\lambda_1h_1(1+h_2l_2) + \lambda_2h_2(1+h_1l_1)} -\right.$$

$$-\frac{z^2}{2\lambda_2} + \sum_{m=1}^{\infty}\frac{J_0\left(\mu_m\frac{r}{R}\right)}{\gamma_m^2 J_0(\mu_m)}\Bigg\{\Bigg[\Big\langle\left(1-\frac{\lambda_1}{\lambda_2}\right)(\gamma_m\sinh\gamma_m l_1 + h_1\cosh\gamma_m l_1)(\gamma_m\cosh\gamma_m l_2 + h_2\sinh\gamma_m l_2) -$$

$$-h_2(\gamma_m\cosh\gamma_m l_1 + h_1\sinh\gamma_m l_1) - h_1(\gamma_m\cosh\gamma_m l_2 + h_2\sinh\gamma_m l_2)\Big\rangle\cosh\gamma_m z +$$

$$+\Big\langle\left(\frac{\lambda_1}{\lambda_2}-1\right)(\gamma_m\sinh\gamma_m l_1 + h_1\cosh\gamma_m l_1)(\gamma_m\sinh\gamma_m l_2 + h_2\cosh\gamma_m l_2) -$$

$$-h_2\frac{\lambda_1}{\lambda_2}(\gamma_m\sinh\gamma_m l_1 + h_1\cosh\gamma_m l_1) + h_1(\gamma_m\sinh\gamma_m l_2 + h_2\cosh\gamma_m l_2)\Big\rangle\sinh\gamma_m z\Bigg]\times$$

$$\times\Big[\lambda_1(\gamma_m\sinh\gamma_m l_1 + h_1\cosh\gamma_m l_1)(\gamma_m\cosh\gamma_m l_2 + h_2\sinh\gamma_m l_2) +$$

$$+\lambda_2(\gamma_m\cosh\gamma_m l_1 + h_1\sinh\gamma_m l_1)(\gamma_m\sinh\gamma_m l_2 + h_2\cosh\gamma_m l_2)\Big]^{-1} + \frac{1}{\lambda_2}\Bigg\}\Bigg), \quad 0 \le z \le l_2. \tag{24}$$

The solution obtained has a rather complicated form. To assure ourselves that the result is accurate, we analyze it. If we set $l_1 = l_2 = l$, $\lambda_1 = \lambda_2 = \lambda$, and $h_1 = h_2 = h$ in Eqs. (23) and (24), then for T_1 and T_2 we obtain

$$T_1 = T_2 = T = \frac{2Q_R}{\lambda R}\left(\frac{l\left(1+\frac{hl}{2}\right)}{h} - \frac{z^2}{2} + \sum_{m=1}^{\infty}\frac{J_0\left(\mu_m\frac{r}{R}\right)}{\gamma_m^2 J_0(\mu_m)}\left(1 - \frac{h\cosh\gamma_m z}{\gamma_m\sinh\gamma_m l + h\cosh\gamma_m l}\right)\right). \tag{25}$$

Expression (25) is in complete agreement with the equation determining the stationary temperature of a homogeneous cylinder which is located symmetrically about the plane $z = 0$ and is heated from the side surface by a constant heat flux with density Q_R and which has the same heat-transfer coefficients on its ends.

Now we consider the satisfaction of the stationarity condition. It is obvious that in the stationary state the total quantity of heat lost to the ends of the cylinders must be equal to that of heat received through the side surface:

$$2Q_R(l_1 + l_2) = R(\alpha_1 T_1(r, -l_1) + \alpha_2 T_2(r, l_2)). \tag{26}$$

Let us now set $z = -l_1$ and $z = l_2$ in Eqs. (23) and (24) respectively and consider the terms before the sum signs (corresponding to the zero root in the equation $J_1(\mu) = 0$) by substituting them into the right-hand side of Eq. (26). Because of the unwieldiness of arithmetic calculations, we give only their final stage:

$$R\,(\alpha_1 T_1\,(r, -l_1) + \alpha_2 T_2\,(r, l_2)) = 2Q_R \frac{(l_1 + l_2)\,(\lambda_1 h_1\,(1 + h_2 l_2) + \lambda_2 h_2\,(1 + h_1 l_1))}{\lambda_1 h_1\,(1 + h_2 l_2) + \lambda_2 h_2\,(1 + h_1 l_1)} = 2Q_R\,(l_1 + l_2)\,. \tag{27}$$

The result obtained coincides with the left-hand side of Eq. (26), thus corresponding to the satisfaction of the condition of energy balance. To determine the value of the contribution of the terms under the sum sign in Eqs. (23) and (24) to the heat flux, it is necessary to integrate these terms over r with account for the corresponding heat-transfer coefficients. Each term of the sum will represent the product of a certain constant by the integral of the form $\int_0^R rJ_0\left(\mu_m \frac{r}{R}\right) dr$. Having introduced the notation $z = r/R$, with the formula $\int_0^1 zJ_0(\mu_m z)\,dz = \frac{1}{\mu_m} J_1(\mu_m)$ [3, 5] taken into account, we obtain $\int_0^R rJ_0\left(\mu_m \frac{r}{R}\right) dr = 0$, since μ_m are the roots of the equation $J_1(\mu) = 0$. Thus, the heat flux from the ends of the cylinders is completely determined by the terms before the sum signs in Eqs. (23) and (24). As the numerical calculations show, the terms under the sum signs give an insignificant contribution to the total temperature and play the role of correcting terms on which the temperature distribution over the radius depends. In the cases where it is necessary to know only the magnitude of the heating and a high accuracy in determining the spatial temperature-field distribution is not required (for example, in diffusion welding of small-size products), it will suffice to use the terms before the sum signs in Eqs. (23) and (24), which have, in addition, a relatively simple form (second-degree polynomials).

NOTATION

R, radius of the cylinders; l_1 and l_2, lengths of the cylinders; λ_1 and λ_2, thermal-conductivity coefficients; α_1 and α_2, heat-transfer coefficients; h_1 and h_2, reduced heat-transfer coefficients; Q_R, surface power of the heat flux; T_1 and T_2, temperatures of the cylinders; $\bar{T}_1$ and $\bar{T}_2$, temperatures of the cylinders in the domain of Hankel transforms; J_0 and J_1, Bessel functions of the first kind and of zero and first orders; μ_m and γ_m, parameters of the finite Hankel transform; z and r, variables of integration.

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