Discrete Mathematics 84 (1990) 193–196
North-Holland

COMMUNICATION

PSEUDODOMISHOLD GRAPHS

A.A. CHERNYAK
Institute of problems of machine reliability, Academy of Sciences of BSSR, Minsk, USSR, 220732

Zh.A. CHERNYAK
Radiotechnical Institute, Minsk, USSR, 220013

Communicated by C. Benzaken
Received 5 July 1989
Revised 21 March 1990

A graph G is called pseudodomishold if there are nonnegative real numbers $c_1, \ldots, c_n, t$ (not all of them zero) such that for every subset U of VG and for its characteristic zero-one vector $(e_1, \ldots, e_n)$ the implications

$$(t < c_1e_1 + \cdots + c_ne_n) \Rightarrow (U \text{ is a dominating set of } G),$$
$$(t > c_1e_1 + \cdots + c_ne_n) \Rightarrow (U \text{ is not a dominating set of } G)$$

are true.

A pseudodomishold graph is called a hereditary one (briefly, HP-graph) if every its induced subgraph is also pseudodomishold.

In this note we describe the structure of HP-graphs in terms of the composition of simplest components and minimal forbidden induced subgraphs. This result gives rise to a simple recurrence formula and estimates for the number u_n of n-vertex HP-graphs. It also implies the linear-time algorithms recognizing HP-graphs and their degree sequences.

Up to now the various classes of P-threshold graphs, namely threshold, pseudothreshold and boxthreshold, have been thorough studied. A number of structure characterizations, recognition algorithms, classification and enumeration aspects concerning P-threshold graphs can be found in [1–9]. There has been little knowledge of pseudodomishold graphs. Unlike thresholdness, pseudothresholdness and domisholdness in definition of which a separating hyperplane plays a principal role, pseudodomisholdness is not a hereditary property, i.e. it is not preserved in proceeding to some induced subgraph. The aim of this communication is to give description and enumeration of hereditary pseudodomishold graphs (briefly, HP-graphs). It is a rather wide class containing the ones of domishold and threshold graphs [1–2].

The main result is Theorem 1 characterizing HP-graphs in terms of the composition of the simplest components and minimal forbidden induced subgraphs. It, in turn, gives rise to a simple recurrence formula and estimates for the number u_n of n-vertex HP-graphs (Corollaries 1–2). Theorem 1 also implies an easy guessed linear-time algorithms recognizing HP-graphs and their degree sequences. It is interesting to compare the estimates for u_n with the asymptotics for the number d_n of n-vertex domishold graphs given in Corollary 3.

All graphs considered are finite, undirected, without loops and multiple edges. VG is the vertex set of a graph G and EG is its edge set. A n-vertex graph G is called pseudodomishold if there are nonnegative real numbers $c_1, \ldots, c_n, t$, not all of them equal zero, such that for every subset U of VG we have

$$\begin{aligned} &(c_1e_1 + \cdots + c_ne_n > t) \Rightarrow (U \text{ is dominating}),\\ &(c_1e_1 + \cdots + c_ne_n < t) \Rightarrow (U \text{ is not dominating}) \end{aligned} \tag{1}$$

where $(e_1, \ldots, e_n)$ is the characteristic vector of U.

If the first inequality in (1) were substituted by the nonstrict inequality $c_1e_1 + \cdots + c_ne_n \geqslant t$ we would have the definition of a domishold graph [1].

Notation. $\bar{G}$ is a graph being complementary to G. K_n is a n-vertex complete graph. M is the set of graphs H such that either any vertex degree in H exceeds $|VH| - 4$ (i.e. all connected components of $\bar{H}$ are only chains or cycles) or H is a 5-vertex chain. t_n is the number of n-vertex graphs from M not containing dominating vertices, isolated vertices and 2-vertex edgeless dominating sets.

So called *Forbidden* configurations are shown in *Fig. 1*: broken lines indicate non-edges.

We will write $G(A, B)$ if the fixed partition $VG = A \cup B$ is meant $(A \cap B = \emptyset)$. One of the parts A, B can be empty. If $G = \bar{K}_n$ then Q_n will denote $G(VG, \emptyset)$. If $G = K_1$ then O_1 will denote $G(\emptyset, VG)$. If G is a 4-vertex chain then P_4 will denote the graph $F(A, B)$ where $|A| = |B| = 2$ and B consists of vertices with degrees being equal to 1.

The composition $\circ$ is defined as follows [8]: given $G(A, B)$ and H, $VH \cap VG = \emptyset$ then

$$G(A, B) \circ H = G \cup H \cup K_{A,VH}$$

where $\cup$ is the sign of the union of graphs, $K_{A,VH}$ is the complete bipartite graph with parts A and VH.

Theorem 1. *The following assertions are equivalent*:

(i) *G is a* HP-*graph.*

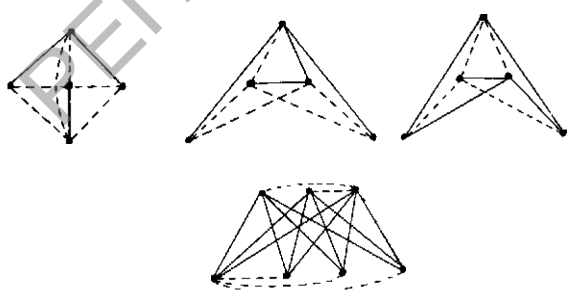

Fig. 1.

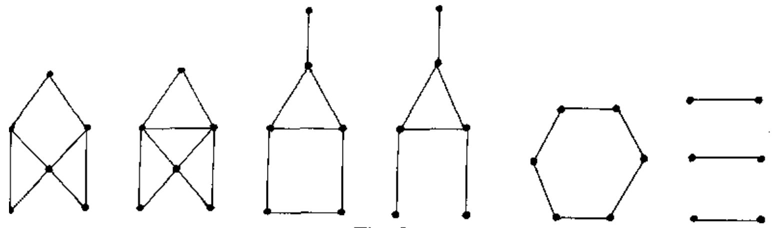

Fig. 2.

(ii) *G contains neither forbidden configurations shown in Fig. 1 nor induced subgraphs shown in Fig. 2.*

(iii) *G can be represented in the form*

$$G = X_1 \circ \cdots \circ X_m \circ H, \; m \geqslant 0,$$

where $H \in M$, $X_i \in \{Q_1, Q_2, O_1, P_4\}$.

Corollary 1.

$$u(x) = \frac{t(x) + x - x^3}{-x^4 + x^3 - x^2 - 2x + 1}$$

where $u(x) = \sum_{n=0}^{\infty} u_n x^n$, $t(x) = \sum_{n=1}^{\infty} t_n x^n$.

Corollary 2. *For* $n \geqslant 6$

$$(2,32)^{n-1} < u_n < (2,37)^{n-1}.$$

Corollary 3. *If* d_n *is the number of n-vertex domishold graphs then*

$$d(x) = \sum_{n=1} d_n x^n = \frac{-x^3 + x}{x^3 - x^2 - 2x + 1}$$

and

$$d_n \sim cr^n$$

where r is the maximum (in absolute value) root of the equation $x^3 - 2x^2 - x + 1 = 0$ $(r \approx 2,24)$.

Problem. What is the complexity of recognizing pseudodomishold (not necessary hereditary) graphs?

Acknowledgements

The authors are very grateful to referees and C. Benzaken for their valuable comments and corrections.

References

[1] C. Benzaken and P.L. Hammer, Linear separation of dominating sets in graphs, Ann. Discrete Math. 3 (1978) 1-10.

[2] V. Chvatal and P.L. Hammer, Aggregation of inequalities in integer programming, Ann. Discrete Math. 1 (1977) 145-162.

[3] P.L. Hammer, T. Ibaraki and B. Simeone, Threshold sequences, SIAM J. Algebraic Discrete Methods 2 (1981) 39-49.

[4] P. Marchioro and A. Morgana, Structure and recognition of domishold graphs, Discrete Math. SO (1984) 239-251.

[5] U.N. Peled and B. Simeone, Boxthreshold graphs. J. Graph Theory 8 (1984) 331-345.

[6] U.N. Peled, Threshold graph enumeration and series-product identities, Proc. 11th S.-E. Conf. on Combin., Graph Theory and Comput. (1980) 735-738.

[7] Ruch and I. Gutman, The branching extent of graphs. J. Combin. Inform. System Sci. 4 (1979) 285-295.

[8] R.I. Tyshkevich and A.A. Chernyak, Decomposition of graphs. Cybernetics 21 (2) (1985) 231-242.

[9] R.I. Tyshkevich and A.A. Chernyak, Boxthreshold graphs: structure and enumeration. 30th Intern. Wiss. Koll, TH Ilmenau, "Graphen und Netzwerke—Theorie und Anwendungen" (1985) 119-121.