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Note

Note on complexity of computing the domination of binary systems

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Abstract

The problem of computing the domination of a coherent binary system all minimal paths sets of which have equal cardinality k ($k > 1$), is proved to be #P-complete. Some corollaries are given.

1. Introduction

The domination theory plays an important part in the study of reliability. Satyanarayana and Prabhakar [13] were the first who defined the domination and evaluated it for directed (s,t)-graphs. Starting with their paper, a sequence of papers has succeeded in characterizing the domination of a system in a number of reliability contexts. Particularly, as shown in [2, 12] the domination theory provides the bases for selecting optimal pivoting strategies. Relations between the domination and matroids are developed in [8] to extend some important domination results to reliability of more general systems. In [7] the domination is interpreted as a partial derivative of the reliability polynomial for studying a length-criterion rooted communication problem. In [3–6] the domination theory is extended to monotone (S,t)-graphs, including some well-known network reliability models as special cases.

In this note we prove (Theorem 1) that the problem of computing the domination for binary coherent systems all minimal paths of which have the fixed cardinality k, $k > 1$, is #P-complete. (Note that the #P-complete class defined in [14] contains problems which are at least as hard as NP-complete ones.) This implies the #P-completeness of the similar problem for monotone (S,t)-graphs (Corollary 1).

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It is interesting to note that in a number of cases such as rooted directed graphs [11, 13], k-out-of-n systems [2], monotone (S,t)-graphs with all vertex functions being symmetric [4], the domination is expressed in a simple analytic form and, hence, can be evaluated efficiently. As for binary coherent systems, the problem of computing the domination turns out to be, in a sense, not more difficult than that of computing the reliability (Proposition 1). This implies, in particular, the polynomial solvability of computing the domination of shellable, regular and threshold systems (Corollaries 2 and 3).

2. Preliminaries

Let $S=\{1,2,\dots,n\}$ be a set of elements and $P=\{P_1,\dots,P_m\}$ be a family of minimal paths where $P_i \subseteq S$, $P_i \not\subseteq P_j$ for $i \neq j$ and $S=\bigcup_{i=1}^{m} P_i$. For a $(0,1)$ n-vector $x=(x_1,\dots,x_n)$ we define the supp(x) as the set $\{j\colon j=1,\dots,n,\ x_j=1\}$. The vector x is called a characteristic vector of a set $A \subseteq S$ if $\mathrm{supp}(x)=A$. Let

$$f(x)=\begin{cases} 1, & \text{if } x \text{ is a characteristic vector of a set } A \\ & \text{containing } P_i \text{ for some } i; \\ 0, & \text{otherwise.} \end{cases}$$

Then f is called the structure function of the pair (S,P). Obviously, f is a monotone boolean function. The triple (S,P,f) is called a coherent system (abbreviated $[S,P]$). The coherent system is k-uniform if $|P_i|=k$ for $i=1,\dots,m$. We call a vector x an operative point (OP) of the function f if $f(x)=1$.

A formation of S is a set of minimal paths whose union is S. It is odd (even) if the number of minimal paths is odd (even). The domination $d(P)$ of $[S,P]$ is the number of odd formations of S minus the number of even formations of S.

Given the rational n-vector $\overline{p}=(p_1,\dots,p_n)$, $0<p_i<1$, $i=1,\dots,n$, we suppose that every element i of S is subject to random failure, independently of others and with the probability $q_i=1-p_i$. Then reliability $h((S,P,f),\overline{p})$ of the system (S,P,f) is the probability that there is at least one minimal path of P consisting of operative elements. We also need some other expression of h. To this end, we define the probability weight prob(x) of a vector x as follows:

$$\mathrm{prob}(x_i)=\begin{cases} p_i, & \text{if } x_i=1, \\ q_i, & \text{if } x_i=0, \end{cases}$$

$$\mathrm{prob}(x)=\prod_{i=1}^{n}\mathrm{prob}(x_i).$$

Now the value $h((S,P,f),\overline{p})$ can be determined as the sum of probability weights of all OPs of f.

We explore the computational complexity of counting problems in the manner proposed in [14]. The #P-class is defined to be the set of rational functions that can be computed by counting the number of accepting computations of some nondeterministic Turning machine of polynomial complexity. We say that a function f is polynomially reducible to a function g if there exists an algorithm which, for input z, evaluates $f(z)$ with a number of elementary operations and evaluations of g that is polynomial in the length of z. A function f is called #P-complete if (a) f is in #P and (b) every function g in #P can be reduced to f by a polynomial time reduction. In what follows we will not distinguish between functions and corresponding counting problems.

3. Main results

We define the size of a rational number r to be the value $\text{size}(r) = \log(u) + \log(v)$ when r is presented as a fraction u/v in the lowest terms. The size of a rational n-vector $\bar{r} = (r_1, \ldots, r_n)$ is then defined as $\text{size}(\bar{r}) = \max\{\text{size}(r_i)\colon i = 1, \ldots, n\}$.

Lemma 1 (Valiant [14]). (i) *If $g(x)$ is an nth degree polynomial with rational coefficients and its value is known at each of the distinct rational points $x_1, \ldots, x_{n+1}$ then the coefficients of g can be deduced in time polynomial in n and the maximum of* $\text{size}(x_i)$, $\text{size}(g(x_i))$, $i = 1, \ldots, n+1$.

(ii) *Let*

$$g(x) = \sum_{i=1}^{n} a_i x^i (1-x)^{n-i}$$

be a polynomial with nonnegative integer coefficients bounded by A. If the value $g(x_0)$ is known at a rational point x_0 and $0 < x_0 \leqslant A^{-2}$ then the coefficients a_i can be deduced in time polynomial in n, $\text{size}(A)$ *and* $\text{size}(x_0)$.

Theorem 1. *The following counting problem is #P-complete*:
DOMINATION OF K-UNIFORM COHERENT SYSTEM (abbreviated DS(k))
Input: *k-uniform coherent system $[S,P]$, k is fixed, $k > 1$.*
Output: *signed domination $d(P)$.*

Proof. Obviously, the counting problem DS(k) is in #P. We will show it is #P-complete. First, let $[S,P]$ be any coherent system. As proved in [8],

$$d(P) = \sum_{r=1}^{n} (-1)^{n-r} A_r \tag{1}$$

where A_r is the number of r-element sets of S containing minimal paths. It follows that

$$d(P) = \sum_{r=1}^{n} (-1)^{n-r} \left(\binom{n}{r} - B_r \right) = \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} + \sum_{r=1}^{n} (-1)^{n-r+1} B_r$$

where $\binom{n}{r}$ is the binomial coefficient, B_r is the number of r-element sets of S not containing minimal paths. As

$$\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} = 0,$$

we can transform the above equality into the following one:

$$d(P)+(-1)^n=\sum_{r=1}^{n}(-1)^{n-r+1}B_r$$

or

$$(-1)^{n-1}d(P)-1=\sum_{r=1}^{n}(-1)^r B_r.$$

Denote by $\mathrm{md}(P)$ the left part of the last equality.

The following counting problem is #P-complete [9]:

INDEPENDENT SET (abbreviated IS)

Input: graph G with a vertex set VG and an edge set EG.

Output: number of independent sets of G, where $I\subseteq \mathrm{VG}$ is called independent if, for all $u,v\in I$, $(u,v)\notin \mathrm{EG}$.

We will prove that IS is polynomially reducible to the counting problem DS(2). Given an instance $G=(\mathrm{VG},\mathrm{EG})$ of IS, let $S=\mathrm{VG}$ and $P=\mathrm{EG}$. Then, $[S,P]$ is a coherent system, and the independent sets of G are exactly the subsets of S not containing any minimal paths. Hence, the number of independent sets of G is equal to $1+\sum_{r=1}^{n}B_r$.

Let $\mathrm{VG}=\{v_i\colon\ i=1,\dots,n\}$. For each t, $1\leqslant t\leqslant n+1$, we construct a graph G_t with the vertex set $\mathrm{VG}_t=\{v_{ij}\colon\ i=1,\dots,n,\ j=1,\dots,t\}$ and the edge set EG_t such that $(v_{ij},v_{st})\in \mathrm{EG}_t$ if and only if $((i=s)$ or $(v_i,v_s)\in \mathrm{EG})$.

Every independent set U' of G_t has the property that

$$|U'\cap\{v_{ij}\colon\ j=1,\dots,t\}|\leqslant 1\quad \text{for } i=1,\dots,n. \tag{2}$$

Each independent set U of G induces a class $I(U)$ of independent sets U' of G_t by the following way:

$$v_{ij}\in U' \quad \text{for some } j \text{ if and only if } v_i\in U.$$

Obviously, the cardinality of $I(U)$ is $t^{|U|}$ and classes $I(U)$ cover all independent sets of G_t because of (2). It follows that the number of r-element independent sets of G_t is $B_r\cdot t^r$.

Now let $S_t=\mathrm{VG}_t$, $P_t=\mathrm{EG}_t$ and $[S_t,P_t]$ be a coherent system. Then

$$\mathrm{md}(P_t)=(-1)^{nt-1}d(P_t)-1=\sum_{r=1}^{n}(-t)^r B_r.$$

Hence, applying Lemma 1(i) to the polynomial

$$g(x)=\sum_{r=1}^{n}(-x)^r B_r,$$

we can evaluate the numbers B_r and, in particular the value $\sum_{r=1}^{n}B_r$, in time polynomial in n and maximum of the values $\mathrm{size}(d(P_t))$. That solves the counting problem IS in polynomial time. The reduction follows.

The polynomial reduction DS(k) to DS($k+1$) is becoming evident if we consider the coherent system $[S\cup\{z\},P']$ obtained from $[S,P]$ by adding a new element z to each of the minimal paths P_i, i.e. $P' = \{P_1 \cup \{z\}, \ldots, P_m \cup \{z\}\}$. It is clear that $d(P) = d(P')$. This completes the proof of the theorem. $\square$

Proposition 1. *If the reliability of a coherent system (S, P, f) can be computed in time polynomial in n and m, then the same is valid for computing the domination of that system.*

Proof. Let A_r be the number of OPs of f having r unit components. Choose p_0 to be equal to $1/2^{2n}$. Obviously,

$$h([S,P], p_0) = \sum_{r=1}^{n} A_r p_0^r (1 - p_0)^{n-r}.$$

Keeping in mind that $A_r \leqslant 2^n$, we conclude from Lemma 1(ii) that numbers A_r can be computed in time polynomial in n and m. Now the proposition follows from formula (1). $\square$

4. Corollaries

Now we give the definition of a monotone (S,t)-graph. Let $G(V,E)$ be a directed graph with source vertices S and a sink vertex t. Given a vertex $v \in V \backslash S$, it is assigned to v a set th(v) of subsets (so-called threshold kits) of the set Adj(v, G) consisting, in turn, of vertices u such that $(u, v) \in E$. In fact, the signal passability across a vertex v is carried out in accordance with a monotone Boolean function with the set th(v) as that of its prime implicants (strict definitions were presented in [6]).

A local formation of a vertex $v \in V \backslash S$ is defined to be a subset of threshold kits from th(v) whose union is Adj(v, G). It is odd (even) if the number of kits in it is odd (even). The local domination $d(v, G)$ is the number of odd local formations of v minus the number of even local formations of v. We set $d(v, G) = 1$ for every $v \in S$. It was proved in [6] that the domination of an acyclic monotone (S,t)-graph is equal to the product of all its local dominations.

Corollary 1. *The problem of computing the domination of monotone (S,t)-graphs is #P-complete.*

Proof. Let (S, P, f) be a 2-uniform coherent system. Define a degenerate monotone (S,t)-graph $G(V,E)$ to have the vertex set $V = S \cup \{t\}$ and the arc set $E = \{(i, t){:}\ i = 1, \ldots, n\}$. Let the set th$(t)$ be the same as P. In this case $d(P) = d(t, G) = d(G)$, which proves the corollary. $\square$

A shelling for a coherent system $[S,P]$ is an ordering of the paths of P such that the discarded set $\{P_1 - P_k, \ldots, P_{k-1} - P_k\}$ consists of single element sets for all $k = 1, \ldots, m$. The system $[S,P]$ is shellable if there exists a shelling for it.

Our Proposition 1 and Proposition 1 of [1] imply

Corollary 2. *If a shelling for $[S,P]$ can be found in polynomial time then the domination of $[S,P]$ can be computed in time polynomial in n and m.*

We say that X is a left shift of X' if $\operatorname{supp}(x)=\{i_1,\dots,i_r\}$, $\operatorname{supp}(x')=\{j_1,\dots,j_r\}$ and $i \leqslant j_1,\dots,i_r \leqslant j_r$. A coherent system (S,P,f) is called regular (2-monotonic) if there exists a renumbering of variables of f such that the set OPs of f is closed under left shifts.

Regular systems are shellable [1]. This implies

Corollary 3. *The domination of a regular coherent system (S,P,f) can be computed in time polynomial in n and m.*

Let $G=(V,E)$ be a finite r-hypergraph with the vertex set V and the edge set E of subsets of V, each of size r. The r-hypergraph G is called threshold if G admits a numbering of its vertices such that for any v_i, v_j $(i<j)$ and any subset $A \subseteq V$ of cardinality $r-1$, $\{v_j \cup A\} \in E$ implies $\{v_i \cup A\} \in E$ (see [10]).

Corollary 4. *The following counting problems are polynomially solvable in threshold r-hypergraphs: counting the number of independent sets (vertex covers) of a fixed cardinality, counting maximum (minimum) cardinality independent sets (vertex covers).*

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