



Interchange Theorems for Hypergraphs and Factorization of Their Degree Sequences

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The aim of this paper is to unify interchange theorems and extend them to hypergraphs. To this end sufficient conditions for equality of the l_1 -distance between equivalence classes and the l_1 -distance between corresponding order-type functions are provided. The generality of this result is demonstrated by a number of new corollaries concerning the factorization and the switching completeness of degree sequences of graphs and hypergraphs.

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INTRODUCTION

Interchange theorems based on simple switching operations for jumping from one graph to another provide constructive techniques for obtaining important results on graphs and $(0,1)$ matrices with invariant characteristics. The idea of unifying interchange theorems and extending them to the class \mathcal{L}_r of all hypergraphs with edge multiplicity at most r is the aim of this paper.

The main instrumental result disclosing the common combinatorial nature of interchange theorems is given in Section 1. For this, on the set of integer-valued functions defined on a disjoint union of finite sets, we introduce a double shift operation, consisting of two symmetric transformations used earlier in [3, 15, 21] for obtaining computable bounds on graph reliability efficiently. This operation defines an equivalence relation on the set of functions: two functions are equivalent if one can be transformed to another by a sequence of double shifts. To each equivalence class corresponds an order-type function which is invariant under double shifts. Theorem 1.1 provides sufficient conditions for equality of the l_1 -distance between equivalence classes and the l_1 -distance between corresponding order-type functions.

The general character of Theorem 1.1 is demonstrated in Section 2 which contains a number of applications of this result:

- (1) The concept of interchange is extended to hypergraphs with a fixed partition of the vertex set. Interchange theorems are deduced for hypergraphs and r -graphs, the latter generalizing corresponding results from [4, 5].
- (2) Criteria for the factorization of vertex and edge degree sequences of hypergraphs are given.
- (3) The problem of finding the switching-complete characteristics of graphs arises in connection with graph generation algorithms [6, 12]. Several switching-complete properties concerning the connectedness of simple graphs were given in [2, 8, 23, 24]. It was proved in [7, 10, 11, 13, 20] that degree sequences are switching-complete parameters in the class of ordinary graphs. Here this result is substantially strengthened. Namely, the switching completeness of edge degree sequences is established in the general class of r -multihypergraphs. A similar result for vertex degree sequences is shown to be valid only for multihypergraphs and r -graphs. It should be noted that degree sequences remain, to our knowledge, unique numerical parameters whose switching completeness is justified in subclasses of hypergraphs.

1. MAIN RESULT

Let $V = \{v_1, \dots, v_n\}$ be a set of distinct elements and let $V = R_1 \cup \dots \cup R_m$ be a fixed partition \mathcal{L} of V . For a set $U \in 2^V$ let $t_i = |U \cap R_i|$, $i = 1, \dots, m$. The vector (t_1, \dots, t_m) is called the *order type* of U . Denote by $T[t_1, \dots, t_m]$ all subsets of V having the order type (t_1, \dots, t_m) .

We call the mappings

$$f : 2^V \rightarrow \{0, 1, \dots, r\}$$

r-functions. The set of all U such that $f(U) \neq 0$ is called the *support* of f and denoted by $\text{supp}(f)$. The *r*-function f is called *degenerate* if its support consists of 2-element subsets of V having the same order type. We write $f \geq g$ if $f(U) \geq g(U)$ for any $U \in 2^V$.

Set $\mathcal{P}(v_i) = \{U : v_i \in U, U \in 2^V\}$, and denote by $d(i, f)$ the sum $\sum_{U \in \mathcal{P}(v_i)} f(U)$. We call an *r*-function f *regular* (with respect to \mathcal{L}) if

$$v_i, v_j \in R_s \text{ implies } |d(i, f) - d(j, f)| \leq 1.$$

Two *r*-functions f and g are called *consistent* (with respect to \mathcal{L}) if

$$v_i, v_j \in R_s \text{ implies } |d(i, f) - d(i, g) + d(j, g) - d(j, f)| \leq 1.$$

We say that an *r*-function f admits a *forbidden configuration* $[v_i, v_j, U, W]$ if

$$v_i, v_j \in R_s, \quad U, W \in 2^V, \quad v_i, v_j \notin U \cup W, \quad f(U \cup v_i) > 0, \\ f(W \cup v_j) > 0, \quad f(U \cup v_j) < r, \quad f(W \cup v_i) < r.$$

Given a forbidden configuration $[v_i, v_j, U, W]$, the *double shift* of an *r*-function f is the transformation of f into a function $g = \text{shift}[v_i, v_j, U, W] \circ f$ defined as follows:

$$g(U \cup v_i) = \max\{0, f(U \cup v_i) - 1\}, \quad g(W \cup v_j) = \max\{0, f(W \cup v_j) - 1\}, \\ g(U \cup v_j) = \min\{r, f(U \cup v_j) + 1\}, \quad g(W \cup v_i) = \min\{r, f(W \cup v_i) + 1\}, \\ g(Q) = f(Q) \quad \text{for all other } Q \in 2^V.$$

Obviously, double shifts preserve regularity, consistency and degeneracy. (Notice that our shifting operation is not related to the shifting operation used in extremal combinatorics.)

On the set of *r*-functions we define an equivalence relation \sim as follows. $f \sim g$ if and only if there exists a sequence of double shifts transforming f into g . The equivalence class containing f is denoted by $[f]$. Each class $[f]$ is associated with the *order-type function* $h_{[f]}$ defined by

$$h_{[f]}(t_1, \dots, t_m) = \sum_{U \in T[t_1, \dots, t_m]} f(U). \quad (1)$$

$h_{[f]}$ is well-defined because the right-hand side of (1) is invariant under double shifts.

Suppose that $\|\varphi\|$ denotes the l_1 -norm of the function φ defined on $\{s_1, \dots, s_k\}$, i.e.,

$$\|\varphi\| = \sum_{i=1}^k |\varphi(s_i)|.$$

Then the distance ρ between classes $[f]$ and $[g]$ is given by

$$\rho([f], [g]) = \min\{\|\bar{f} - \bar{g}\| : (\bar{f}, \bar{g}) \in [f] \times [g]\}.$$

Throughout the following we omit the brackets in expressions of the form $X \cup \{v\}$, i.e., we write $X \cup v$.

THEOREM 1.1. *Let f and g be consistent r -functions. Suppose that either $r = \infty$ or both f and g are regular or both f and g are degenerate. Then*

$$\rho(\{f\}, \{g\}) = \|h_{[f]} - h_{[g]}\|.$$

PROOF. Let $q = f - g$. Then

$$\begin{aligned} \|q\| &= \sum_{(t_1, \dots, t_m)} \sum_{U \in T[t_1, \dots, t_m]} |q(U)| \geq \sum_{(t_1, \dots, t_m)} \left| \sum_{U \in T[t_1, \dots, t_m]} q(U) \right| \\ &= \sum_{(t_1, \dots, t_m)} |h_{[f]}(t_1, \dots, t_m) - h_{[g]}(t_1, \dots, t_m)| = \|h_{[f]} - h_{[g]}\|. \end{aligned} \quad (2)$$

Therefore

$$\rho(\{f\}, \{g\}) \geq \|h_{[f]} - h_{[g]}\|. \quad (3)$$

Suppose that inequality (3) is strict. Then, in view of (2), there exists a pair of sets (X, Y) of the same order type such that $q(X) > 0$, $q(Y) < 0$. Such pairs will be called *signed* for the couple (f, g) .

Note two obvious facts used in what follows:

If (X, Y) is a signed pair for (f, g) , then

$$f(X) > 0, \quad g(X) < r, \quad f(Y) < r, \quad g(Y) > 0. \quad (*)$$

If a and b are integers, $a < b$, then

$$|a| + |b| - |a + 1| - |b - 1| \geq 0. \quad (**)$$

Among $(\bar{f}, \bar{g}) \in [f] \times [g]$ such that $\|\bar{f} - \bar{g}\| = \rho(\{f\}, \{g\})$, choose a pair (f, g) having a signed pair (U, W) with a large as possible intersection $Z = U \cap W$. Obviously, U and W can be represented as

$$U = Z \cup P \cup v_j, \quad W = Z \cup Q \cup v_h,$$

where $v_j, v_h \in R_s$ for some $1 \leq s \leq m$. Consider two cases.

Case 1. $P = Q = \emptyset$. Since

$$\sum_{X \in \mathcal{P}(v_j)} q(X) = d(j, f) - d(j, g), \quad \sum_{X \in \mathcal{P}(v_h)} q(X) = d(h, f) - d(h, g),$$

in view of the consistency of f and g ,

$$\sum_{X \in \mathcal{P}(v_j \setminus v_h)} q(X) - \sum_{X \in \mathcal{P}(v_h \setminus v_j)} q(X) \leq 1, \quad (4)$$

where $\mathcal{P}(v_i \setminus v_s) = \{X \in 2^V : v_i \in X, v_s \notin X\}$. But $q(U) - q(W) \geq 2$. Hence there exists a Y not containing v_j and v_h such that $q(Y \cup v_j) < q(Y \cup v_h)$. It follows that

$$2r < f(Y \cup v_h) + g(Y \cup v_j) + (r - f(Y \cup v_j)) + (r - g(Y \cup v_h)).$$

As all the summands are nonnegative and do not exceed r on the right-hand side of the inequality, at least three of them are positive. In particular, at least one of the following conditions holds:

$$f(Y \cup v_h) > 0, \quad f(Y \cup v_j) < r; \quad (5)$$

$$g(Y \cup v_j) > 0, \quad g(Y \cup v_h) < r. \quad (6)$$

If (5) holds, then, in view of (*), f admits the forbidden configuration $[v_j, v_h, Z, Y]$. In this case we set

$$f_1 = \text{shift}[v_j, v_h, Z, Y] \circ f, \quad g_1 = g.$$

Otherwise, if (6) holds, then, by (*), g admits the forbidden configuration $[v_j, v_h, Y, Z]$. In this case we let

$$f_1 = f, \quad g_1 = \text{shift}[v_j, v_h, Y, Z] \circ g.$$

Thus, setting $q_1 = f_1 - g_1$, $R = Y \cup v_j$, $T = Y \cup v_h$ (and applying (**)) we have

$$\begin{aligned} \|q\| - \|q_1\| &= |q(U)| + |q(W)| + |q(R)| + |q(T)| \\ &\quad - |q(U) - 1| - |q(W) + 1| - |q(R) + 1| - |q(T) - 1| \\ &= |q(R)| + |q(T)| - |q(R) + 1| - |q(T) - 1| + 2 \geq 2. \end{aligned}$$

But $(f_1, g_1) \in [f] \times [g]$, which contradicts the choice of f and g .

Case 2. $|P| = |Q| > 0$. Note that (4) is valid in this case as well. Let $U_h = Z \cup P \cup v_h$, $W_j = Z \cup Q \cup v_j$. Since $|U_h \cap W_j| > |Z|$, according to the choice of the signed pair (U, W) we have $q(U_h) \leq 0$. Similarly, $|U_h \cap U_j| > |Z|$ and, by the same argument, $q(U_h) \geq 0$. Therefore, $q(U_h) = 0$. Similarly, $q(W_j) = 0$. It follows that

$$(q(U) + q(W_j)) - (q(U_h) + q(W)) \geq 2.$$

By (4) there exists a Y not containing v_j and v_h such that $q(Y \cup v_j) < q(Y \cup v_h)$. So, as in Case 1, at least one of the conditions (5), (6) holds.

First suppose that f and g are degenerate. Then $Z \cup P = \{v_i\}$, $Z \cup Q = \{v_s\}$, $Y = v_t$, $v_t, v_i, v_s \in R_l$ for some $1 \leq l \leq m$. Moreover, either $q(v_t \cup v_j) < 0$ or $q(v_t \cup v_h) > 0$. If $q(v_t \cup v_j) < 0$, then setting $U' = Z' \cup v_t$, $W' = Z' \cup v_t$, $Z' = v_j$, we arrive at Case 1. If $q(v_t \cup v_h) > 0$, then setting $U' = Z' \cup v_t$, $W' = Z' \cup v_s$, $Z' = v_h$, again we are in the conditions of Case 1.

Thus in what follows, we take f and g nondegenerate. Consider two subcases.

Subcase 2.1. $|f(U_h) - f(W_j)| \neq r$. Clearly at least one of the following conditions holds:

$$f(U_h) > 0, \quad f(W_j) > 0; \quad (7)$$

$$f(U_h) < r, \quad f(W_j) < r. \quad (8)$$

Suppose that (7) holds. If, in addition, (5) holds, then by (*) f admits the forbidden configuration $[v_h, v_j, Y, Z \cup Q]$. In this case we set

$$f_1 = \text{shift}[v_h, v_j, Y, Z \cup Q] \circ f, \quad g_1 = g.$$

It follows that

$$\begin{aligned}\|q\| - \|q_1\| &= |q(T)| + |q(R)| + |q(W_j)| + |q(W)| \\ &\quad - |q(T) - 1| - |q(R) + 1| - |q(W_j) - 1| - |q(W) + 1| \\ &= |q(T)| + |q(R)| - |q(T) - 1| - |q(R) + 1| \geq 0.\end{aligned}$$

But $q_1(W_j) = -1 < 0$, $q_1(U) = q(U) > 0$, $|U \cap W_j| > |Z|$, contrary to the choice of (U, W) .

If (6) holds, then $f(U_h) = g(U_h) > 0$, therefore g admits the forbidden configuration $[v_h, v_j, Z \cup P, Y]$. In this case we set

$$f_1 = f, \quad g_1 = \text{shift}[v_h, v_j, Z \cup P, Y] \circ g.$$

It follows that

$$\begin{aligned}\|q\| - \|q_1\| &= |q(T)| + |q(R)| + |q(U_h)| + |q(U)| \\ &\quad - |q(T) - 1| - |q(R) + 1| - |q(U_h) + 1| - |q(U) - 1| \\ &= |q(T)| + |q(R)| - |q(T) - 1| - |q(R) + 1| \geq 0.\end{aligned}$$

But $q_1(U_h) = 1 > 0$, $q_1(W) = q(W) < 0$, $|U_h \cap W| > |Z|$, which contradicts the choice of (U, W) .

Since (7) and (8) are symmetric, this concludes the analysis of Subcase 2.1. Moreover, if $r = \infty$ the theorem is proved.

Subcase 2.2. $|f(U_h) - f(W_j)| = r$. Without loss of generality assume that $f(U_h) = r$, $f(W_j) = 0$. If $g(Y \cup v_j) > g(Y \cup v_h)$, then (6) holds and the arguments of Subcase 2.1 can be applied. Now suppose that $g(Y \cup v_j) \leq g(Y \cup v_h)$ and (5) holds. Since

$$0 = g(Z \cup Q \cup v_j) < g(Z \cup Q \cup v_h), g(Z \cup P \cup v_j) < g(Z \cup P \cup v_h) = r < \infty$$

and g is regular, there exists X such that $g(X \cup v_j) > g(X \cup v_h)$. If, in addition, $q(X \cup v_j) < q(X \cup v_h)$ and Y is replaced by X , then (6) holds and the arguments of Subcase 2.1 can be applied. Suppose that $q(X \cup v_j) \geq q(X \cup v_h)$. Then $f(X \cup v_j) > f(X \cup v_h)$. Obviously, f and g admit the forbidden configurations $[v_j, v_h, X, Y]$ and $[v_j, v_h, X, Z \cup P]$, respectively. Let

$$f_1 = \text{shift}[v_j, v_h, X, Y] \circ f, \quad g_1 = \text{shift}[v_j, v_h, X, Z \cup P] \circ g.$$

Then

$$\begin{aligned}\|q\| - \|q_1\| &= |q(T)| + |q(R)| + |q(U_h)| + |q(U)| - |q(T) - 1| - |q(R) + 1| \\ &\quad - |q(U_h) + 1| - |q(U) - 1| \geq 0.\end{aligned}$$

But $q_1(W) = q(W) < 0$, $q_1(U_h) = q(U_h) + 1 > 0$, $|W \cap U_h| > |Z|$, contrary to the choice of (U, W) . The theorem is proved. \square

We present two examples to show the sharpness of Theorem 1.1.

EXAMPLE 1.1. Let $V = \{1, \dots, 8\}$, $R_i = \{2i - 1, 2i\}$, $i = 1, 2, 3, 4$. Consider two r -functions f and g whose supports are $\{1, 5\}$, $\{1, 6\}$, $\{1, 8\}$, $\{2, 8\}$, $\{3, 5\}$, $\{4, 5\}$, $\{4, 7\}$, $\{4, 8\}$ and $\{1, 5\}$, $\{1, 7\}$, $\{1, 8\}$, $\{2, 5\}$, $\{3, 8\}$, $\{4, 5\}$, $\{4, 6\}$, $\{4, 8\}$, respectively. Clearly f and g are

consistent, but neither degenerate nor regular. Neither of them admits a forbidden configuration. So

$$[f] = \{f\}, [g] = \{g\}, \quad \rho([f], [g]) = \|f - g\| = 8.$$

However,

$$\begin{aligned} h_{[f]}(i, j) &= h_{[g]}(i, j) = 2 && \text{for } i = 1, 2 \text{ and } j = 3, 4, \\ h_{[f]}(i, j) &= h_{[g]}(i, j) = 0 && \text{elsewhere.} \end{aligned}$$

Thus $\|h_{[f]} - h_{[g]}\| = 0 < 8$.

EXAMPLE 1.2. Let $V = \{1, 2, 3, 4, 5, 6\}$, $R_1 = \{1\}$, $R_2 = \{2\}$, $R_3 = \{3, 4\}$, $R_4 = \{5, 6\}$. Consider two 1-functions f and g whose supports are $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 5\}$, $\{2, 6\}$ and $\{1, 3\}$, $\{1, 6\}$, $\{2, 3\}$, $\{2, 4\}$, respectively. Clearly, f and g are regular, but not consistent. Neither of them admits a forbidden configuration. Thus $\rho([f], [g]) = \|f - g\| = 6$. However,

$$h_{[f]}(1, 3) = h_{[f]}(2, 4) = 2, \quad h_{[f]}(1, 4) = h_{[f]}(2, 3) = 1,$$

$h_{[g]}(1, 3) = h_{[g]}(1, 4) = 1$, $h_{[g]}(2, 3) = 2$, and $h_{[f]}$ and $h_{[g]}$ are equal to zero elsewhere. Hence $\|h_{[f]} - h_{[g]}\| = 4 < 6$.

COROLLARY 1.1. Let f and g satisfy the conditions of Theorem 1.1. Then $[f] = [g]$ if and only if $h_{[f]} \equiv h_{[g]}$.

COROLLARY 1.2. Let f and g satisfy the conditions of Theorem 1.1. Then $h_{[f]} \geq h_{[g]}$ if and only if there exists a pair $(\bar{f}, \bar{g}) \in [f] \times [g]$ such that $\bar{f} \geq \bar{g}$.

2. PARTICULAR CASES

Denote by $\mathcal{L}_r(k)$ the set of finite hypergraphs in which each edge has the cardinality k and multiplicity at most r , where $k \in N$, $r \in N \cup \{\infty\}$. Let $\mathcal{L}_r = \bigcup_{k=2}^{\infty} \mathcal{L}_r(k)$. In the usual terminology \mathcal{L}_r , \mathcal{L}_1 , $\mathcal{L}_1(k)$, $\mathcal{L}_r(2)$, $\mathcal{L}_1(2)$ are, respectively, r -multihypergraphs, simple hypergraphs, k -uniform hypergraphs, r -graphs, simple graphs. For $G \in \mathcal{L}_r$ VG and EG denote the vertex set and edge set of G . The degree $d(v, G)$ of a vertex v of G is the number of its edges containing v . G_2 is a spanning subhypergraph of G_1 ($G_2 \leq G_1$) if $VG_1 = VG_2$ and $EG_2 \subseteq EG_1$. If $G_2 \leq G_1$, $VG_2 = \{v_1, \dots, v_n\}$, $l_i = d(v_i, G_2)$, $i = 1, \dots, n$, then G_2 is called an (l_1, \dots, l_n) -factor of G_1 (if $l_i = l$, $i = 1, \dots, n$, then G_2 is called an l -factor of G_1).

For a hypergraph $G \in \mathcal{L}_r$ fix a partition

$$R_1 \cup R_2 \cup \dots \cup R_m \tag{9}$$

of its vertex set where R_i are called *blocks*. Partition (9) is trivial when $m = 1$. We say that edges $U_1, U_2 \in EG$ generate a *forbidden configuration* if the following holds: there exist vertices u_1, u_2 contained in the same block such that $u_i \in U_i$, $u_i \notin U_{i+1}$, the sets $W_i = (U_i \setminus u_i) \cup u_{i+1}$ having a multiplicity not equal to r , $i = 1, 2$ (indices are modulo 2). We define an *interchange* over G , in which U_1 and U_2 generate the forbidden configuration, to be the transformation decreasing by 1 the multiplicity of U_1 and U_2 and increasing by 1 the multiplicity of W_1 and W_2 . If $m = 1$, the above definitions are equivalent to the classic concepts of interchange [1, 13, 20] and forbidden configuration [18, 19].

Now suppose that each block R_i consists of vertices of degree d_i , $i = 1, \dots, m$, where all d_i are distinct. Then (9) is called the *degree partition* for G and the set $\{d_1, \dots, d_m\}$ is the degree

set of G . The degree of an edge U of G is the sequence (a_1, \dots, a_m) such that a_i is the number of vertices of degree d_i contained in U , $i = 1, \dots, m$. The edge (vertex) degree sequence $\text{edeg}(G)$ ($\text{vdeg}(G)$) is the list of edge (vertex) degrees of G . Degree sequences coinciding under an appropriate permutation of their terms are considered to be equal. Note that $\text{edeg}(G)$ uniquely determines $\text{vdeg}(G)$ [9]. Hence the degree sets of hypergraphs G and H are the same provided their edge degree sequences are equal.

If α is a degree sequence of $G \in \mathcal{L}_r(k)$ then α is called *realizable* in $\mathcal{L}_r(k)$ and G is the *realization* of α . Let $\alpha = (a_1, \dots, a_p)$ and $\beta = (b_1, \dots, b_p)$. We say that α is *consistent* with β , if $l \leq a_i - b_i \leq l + 1$ for some integer $l \geq 0$. $|\alpha|$ will denote the length of α .

Let $G \in \mathcal{L}_r$ and let (9) be an arbitrary partition of VG . Define an r -function f corresponding to (9) as follows: for $U \subseteq VG = \{v_1, \dots, v_n\}$ $f(U)$ is the multiplicity of U in G . Clearly, $d(i, f) = d(v_i, G)$ and an operation of double shift over f corresponds to an interchange over G .

Now suppose that (9) is the degree partition for G . Then the order type of an essential element U of f is the degree of the edge U in G , and the order-type function $h_{[f]}(t_1, \dots, t_m)$ gives the number of edges of degree (t_1, \dots, t_m) , thereby determining $\text{edeg}(G)$.

Let (9) be the degree partition for G and H . (Note that the degree sets of G and H are not necessarily the same). We say that $\text{edeg}(G)$ majorizes $\text{edeg}(H)$ if each term (t_1, \dots, t_m) occurs in $\text{edeg}(G)$ at least as many times as in $\text{edeg}(H)$.

Taking into account the preceding comments, from Corollary 1.2 we obtain the following result.

THEOREM 2.1. *Let (9) be the degree partition of $H_1, H_2 \in \mathcal{L}_r$ and let $VH_1 = VH_2$. Then using a sequence of interchanges one can transform H_1, H_2 into G_1, G_2 such that $G_2 \leq G_1$ if and only if $\text{edeg}(H_1)$ majorizes $\text{edeg}(H_2)$.*

Since the regularity of r -functions in Corollary 1.2 is not necessary when $r = \infty$, we have:

THEOREM 2.2. *Let $H_1, H_2 \in \mathcal{L}_\infty$, $VH_1 = VH_2$ and let (9) be a vertex partition of H_1, H_2 and let $m = 1$. Suppose that $\text{vdeg}(H_1)$ is consistent with $\text{vdeg}(H_2)$. Then using a sequence of interchanges one can transform H_1, H_2 into G_1, G_2 such that $G_2 \leq G_1$ if and only if for each k the number of edges of cardinality k in H_1 is not less than that in H_2 .*

THEOREM 2.3. *Let $H_1, H_2 \in \mathcal{L}_r(2)$, $VH_1 = VH_2$ and let (9) be a vertex partition of H_1, H_2 and let $m = 1$. Suppose that $\text{vdeg}(H_1)$ is consistent with $\text{vdeg}(H_2)$. Then using a sequence of interchanges one can transform H_1, H_2 into G_1, G_2 such that $G_2 \leq G_1$.*

PROOF. The r -graphs H_1 and H_2 correspond to degenerate r -functions whose supports have the same order type (2). Because of this the condition of regularity in Corollary 1.2 is not necessary. \square

THEOREM 2.4. *Let H_1 and H_2 be bipartite r -graphs with parts R_1 and R_2 and let (9) be a vertex partition of H_1, H_2 and let $m = 2$. Suppose that $\text{vdeg}(H_1)$ is consistent with $\text{vdeg}(H_2)$. Then using a sequence of interchanges one can transform H_1, H_2 into G_1, G_2 such that $G_2 \leq G_1$.*

Denote by $\mathcal{U}(\alpha, \beta, r)$ the set of $|\alpha| \times |\beta|$ matrices over $\{0, 1, \dots, r\}$ having the prescribed row sum vector α and column sum vector β . The following corollary of Theorem 2.4 is a generalization of the corresponding result from [5].

COROLLARY 2.1. *Let $\mathcal{U}(\alpha_i, \beta_i, r) \neq \emptyset$, $i = 1, 2$ and let $\alpha_1(\beta_1)$ be consistent with $\alpha_2(\beta_2)$. Then there exist matrices $A_i \in \mathcal{U}(\alpha_i, \beta_i, r)$ such that $A_2 \leq A_1$.*

Let x be some structure numerical parameter of a hypergraph G of a subset \mathcal{M} of hypergraphs \mathcal{L}_r with the trivial partition of their vertex sets. An interchange over G is called *conservative* in the class \mathcal{M} with respect to x if it transforms G without changing the value of x into a hypergraph from \mathcal{M} . It is defined the equivalence relation $\overset{x}{\sim}$ on the set \mathcal{M} as follows: $G \overset{x}{\sim} H$ if and only if there exists a sequence of conservative interchanges transforming G into a hypergraph isomorphic to H . The parameter x is *switching complete* in the class \mathcal{M} if any two hypergraphs of \mathcal{M} are contained in the same equivalence class if and only if they have equal x .

Notice that any interchange over G with respect to the degree partition is conservative with respect to the edge degree sequence. Hence, from Corollary 1.1 we have

THEOREM 2.5. *Edge degree sequences are switching complete in the class \mathcal{L}_r .*

COROLLARY 2.2. *Edge degree sequences are switching complete in the class $\mathcal{L}_r(k)$ for any $r \geq 1, k \geq 2$.*

Corollary 2.2 generalizes the corresponding result from [10, 11] for $k = 2$.

THEOREM 2.6. *Vertex degree sequences are switching complete in the class $\mathcal{L}_r(k)$ if and only if either $r = \infty$ or $k = 2$.*

PROOF. The sufficiency follows from Theorems 2.2 and 2.3, for necessity, we find a counterexample in the class $\mathcal{L}_1(3)$ (it can be extended to the case of arbitrary $r \geq 2$ and $k \geq 4$).

Let $VG = \{1, \dots, 14\}$ and let EG consist of all triples $\{i, j, k\}$ such that $i < j < k$ and at least one of the following possibilities holds:

- (a) $i = 1, \quad j \leq 5, \quad k \leq 14;$
- (b) $i \leq 2, \quad j \leq 7, \quad k \leq 12;$
- (c) $i \leq 3, \quad j \leq 6, \quad k \leq 10.$

Set

$$VH = VG$$

$$EH = (EG \setminus \{\{1, 5, 14\}, \{2, 7, 12\}, \{3, 6, 10\}, \{1, 4, 13\}, \{2, 6, 11\}, \{3, 5, 9\}\}) \cup \{\{1, 10, 12\}, \{2, 3, 14\}, \{5, 7, 6\}, \{1, 9, 11\}, \{2, 3, 13\}, \{4, 5, 6\}\}.$$

By definition, $\text{vdeg}(G) = \text{vdeg}(H)$. But G and H are not isomorphic because G does not admit a forbidden configuration while the edges $\{1, 9, 11\}$ and $\{1, 10, 12\}$ generate a forbidden configuration in $H(\{1, 9, 12\}, \{1, 10, 11\} \notin EH)$. \square

In the special case of $k = 2$ Theorem 2.6 implies the corresponding result from [7, 13].

From Theorems 2.2 and 2.3 we have

COROLLARY 2.3. *Let $l \leq l_i \leq l + 1, i = 1, \dots, p$ and let a sequence $\alpha = (a_1, \dots, a_p)$ be realizable in $\mathcal{L}_\infty(k)$ (or $\mathcal{L}_r(2)$). Then α has a realization in $\mathcal{L}_\infty(k)$ (or $\mathcal{L}_r(2)$) containing an (l_1, \dots, l_p) -factor if and only if the sequence $(a_1 - l_1, \dots, a_p - l_p)$ is realizable in $\mathcal{L}_\infty(k)$ (or $\mathcal{L}_r(2)$).*

As for the set $\mathcal{L}_1(2)$, Corollary 2.3 implies the well-known result on the factorization of vertex degree sequences conjectured in [14] and proved in [16, 17].

Let

$$\beta = (\beta_1^{k_1}, \dots, \beta_n^{k_n}), \quad \beta_i = (t_{i1}, \dots, t_{im}), \quad i = 1, \dots, n, \quad t_{ij}, k_s \in N. \quad (10)$$

$\binom{a}{b}$ denotes a binomial coefficient; $\binom{a}{b} = 0$ when $b > a$; $\beta_i^{k_i}$ means that β_i occurs k_i times in β . We need the following

LEMMA 2.1 ([9]). *The sequence (10) is realizable in \mathcal{L}_r with hypergraphs having the degree set $\{d_1, \dots, d_m\}$ if and only if the following conditions hold:*

$$\frac{1}{d_i} \sum_{s=1}^n t_{si} k_s = l_i, \quad \text{where } l_i \text{ is integral, } i = 1, \dots, m$$

$$k_s \leq r \cdot \prod_{i=1}^m \binom{l_i}{t_{si}}, \quad s = 1, \dots, n$$

(it is set that $\infty \cdot 0 = 0$).

Suppose that (10) is realizable in \mathcal{L}_r with a hypergraph having the degree set $\{d_1, \dots, d_m\}$. Then

$$\text{vdeg}(G) = (d_1^{l_1}, \dots, d_m^{l_m}).$$

Let 0_n be the $n \times n$ zero matrix and let I_n be the identity matrix of order n . We also set $B = [t_{ij}]_{m \times n}$,

$$A = \begin{bmatrix} B & 0_n \\ I_n & I_n \end{bmatrix}, \quad x^T = (y_1, \dots, y_n, z_1, \dots, z_n),$$

$$a^T = ((d_1 - p)l_1, \dots, (d_m - p)l_m, k_1, \dots, k_n).$$

THEOREM 2.7. *Let (10) be realizable in \mathcal{L}_r with hypergraphs having the degree set $\{d_1, \dots, d_m\}$. Then (10) has a realization in \mathcal{L}_r containing a p -factor ($p \leq \min d_i$) if and only if the system*

$$Ax = a, \quad x \geq 0 \quad (11)$$

has an integral solution.

PROOF. The necessity: let G be a realization of (10). Suppose that $H, G \in \mathcal{L}_r$, $H \leq G$, $d(v, G) - d(v, H) = p$ for any vertex v . Then

$$\text{vdeg}(H) = ((d_1 - p)^{l_1}, \dots, (d_m - p)^{l_m}),$$

$$\text{edeg}(H) = (\beta_1^{s_1}, \dots, \beta_n^{s_n}),$$

$$s_i \leq k_i, \quad i = 1, \dots, n.$$

Besides,

$$(d_i - p)l_i = \sum_{j=1}^n t_{ji} s_j, \quad i = 1, \dots, m.$$

Clearly $(s_1, \dots, s_n, k_1 - s_1, \dots, k_n - s_n)^T$ is a desired solution of (11).

The sufficiency: let x^T be an integral solution of (11). Then

$$(d_i - p)l_i = \sum_{j=1}^n t_{ji} y_j, \quad i = 1, \dots, m, \quad y_s \leq k_s, \quad s = 1, \dots, n.$$

By Lemma 2.1,

$$k_s \leq r \cdot \prod_{i=1}^m \binom{l_i}{t_{si}}.$$

Hence, again by Lemma 2.1, the sequence

$$\gamma = (\beta_1^{y_1}, \dots, \beta_n^{y_n})$$

is realizable in \mathcal{L}_r with hypergraphs having the degree set $\{d_1 - p, \dots, d_m - p\}$. Since β majorizes γ , the rest follows from Theorem 2.1. \square

The conditions of realizability of vertex degree sequences in $\mathcal{L}_\infty(k)$ and $\mathcal{L}_r(2)$ are effectively verified [4, 7]. Because of this the criterion given in Corollary 2.3 is effectively verified as well. Unfortunately, polynomial solvability of the corresponding problem for edge degree sequences, in view of Theorem 2.7, appears to be unlikely apart from some special cases. One of them is given in the following

COROLLARY 2.4. *Let B be the incidence matrix of a bipartite graph. Then the problem of existence of a realization of (10) in $\mathcal{L}_1(2)$ containing a p -factor is solvable in polynomial time.*

PROOF. According to [22], the matrix B is unimodular, as is the matrix A defined above. It implies polynomial solvability of the problem of existence of integral solution of (11) [22]. The rest follows from Theorem 2.7. \square

A hypergraph G from $\mathcal{L}_1(k)$ is called the T_3 -threshold [19] if there exists a numbering of its vertices such that for any $v_i, v_j \in VG, i < j$, and any $(k-1)$ -element subset $U \subseteq VG$ not containing v_i and v_j , the following sentence is true:

$$(\{v_j \cup U\} \in EG) \Rightarrow (\{v_i \cup U\} \in EG).$$

LEMMA 2.2 ([19]). *A hypergraph G from $\mathcal{L}_1(k)$ is T_3 -threshold if and only if G does not contain any forbidden configuration (under the trivial partition of VG).*

From Lemma 2.2 and Theorem 2.5 we have

COROLLARY 2.5. *T_3 -threshold hypergraphs are uniquely determined by their edge degree sequences.*

COROLLARY 2.6. *Each k -uniform T_3 -threshold hypergraph is uniquely determined by its vertex degree sequence if and only if $k = 2$.*

PROOF. T_3 -threshold hypergraphs are threshold graphs [19] which are known to be uni-graphs [18]. The rest follows from the proof of Theorem 2.6 where one of the constructed hypergraphs is T_3 -threshold (by Lemma 2.2). \square

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